Mountain waves in a turbulent atmosphere

By LÉON COVEZ, *Faculty of Science, Paris*

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ABSTRACT

The propagation of mountain waves in an atmosphere with a fluctuating parameter of stability is studied. Two cases are considered, the Gaussian, and a discontinuous Markov chain, corresponding to a dry and a wet atmosphere, respectively.

1. Introduction

The use of simple mathematical models in dynamic meteorology has given some results in the study of the general circulation (barotropic planetary waves) and mountain waves.

Barotropic waves are related to the problem of forecasting long-term changes in the weather and the problem of elaborating a physico-mathematical description of the general circulation. The numerical experiments differ from each other more with respect to the manner of taking into account energy dissipation due to horizontal and vertical turbulent exchange. Turbulent exchange is usually taken into account by the introduction of a "fictitious viscosity" and as Blinova (1965) has pointed out, this only leads to an attenuation of the process, that is to a climate.

Barrier mountain waves are important in the study of the so-called "middle scale" perturbations in particular cyclones. In most models turbulent exchange is completely neglected and humidity and phase changes are not taken into consideration. In what follows we will discuss the propagation of mountain waves in an atmosphere with an index of refraction which suffers small changes depending upon temperature and humidity conditions. These fluctuations created by the atmospheric turbulence generate a wave dispersion independent of the wave dispersion originated by reflections in the boundaries.

In 1937 and 1941, Dahl, Devik, Bovsheverov and Krassilnikov (Ellison, 1951) made the suggestion that measures of these fluctuations could be a source of information on turbulence.

The mathematical formulation of this problem leads to linear partial differential equations with random coefficients (in time and space). The theory of random equations has been developed in recent times by Bourret (1962), Kraichnan (1962), Bharucha-Reid (1964), Kampé de Fériet (1965), Frisch (1968) and others.

In 1961 Kraichnan showed the equivalence between a linear stochastic equation and a non-linear non-random equation; thus the solution found directly to linear stochastic equations derived from a linearized model of the hydrodynamic equations of motion can throw some light upon the non-linear problem.

The one-dimensional wave equation with Gaussian and Markovian coefficients can be solved exactly (Frisch, 1968).

2. Mathematical model

Scorer reached theoretically the conclusion, that the nature of the airflow over a rugged terrain depends on the parameter

$$l^2 = \frac{g}{u^2}\,\frac{\gamma_a-\gamma}{T} - \frac{1}{u}\,\frac{\partial^2 u}{\partial z^2} \qquad (1)$$

This is a stability parameter; it can in practice be assumed, that

$$l^2 \approx \frac{g}{u^2}\,\frac{\gamma_a-\gamma}{T} \qquad (2)$$

unfortunately until now there has been no experimental investigation of the parameter l^2.

In the case of a dry atmosphere the fluctuations of the parameter l^2 can be described by a distribution with a Gaussian probability law. In the case of a wet atmosphere, condensation can be simulated by a particular kind of Markov process (Chen, 1962). We will consider only fluctuations in space and this is equivalent to taking atmospheric stratification into account without establishing artificial boundaries.

Let us suppose that a uniform air current whose velocity, pressure and density are respectively

$$\bar{u} = \text{const.}, \quad p = \bar{p}(z) \quad \varrho = \bar{\varrho}(z)$$

(z is the vertical coordinate), encounters a mountain given by an equation of the form $z = \zeta(x, y)$. The disturbances in the fields of the pressure, density and velocity components generated by the barrier in the air current respectively by

$$p'(x, y, z) \quad \varrho'(x, y, z) \quad u'(x, y, z)$$

and $$w'(x, y, z)$$

Thus we have

$$u = \bar{u} + u' \quad v = v' \quad w = w' \quad p = \bar{p} + p'$$

and $$\varrho = \bar{\varrho} + \varrho'$$

The problem consists in determining u', v', w', p' and ϱ'. If there is a vertical temperature distribution we have

$$T = T_0 - \gamma z$$

T_0 = temperature at ground level
γ = lapse rate

We will limit ourselves to the stationary case; then, the equations of motion and heat inflow are the following:

$$\left.\begin{aligned} & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\varrho}\frac{\partial p}{\partial x} \\ & u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\varrho}\frac{\partial p}{\partial y} \\ & -\frac{1}{\varrho}\frac{\partial p}{\partial z} - g = 0 \\ & \frac{\partial \varrho u}{\partial x} + \frac{\partial \varrho v}{\partial y} + \frac{\partial \varrho w}{\partial z} = 0 \\ & u\frac{\partial}{\partial x}\left(\frac{p}{\varrho^{\varkappa}}\right) + v\frac{\partial}{\partial y}\left(\frac{p}{\varrho^{\varkappa}}\right) + w\frac{\partial}{\partial z}\left(\frac{p}{\varrho^{\varkappa}}\right) = 0 \end{aligned}\right\} \quad (3)$$

$f = 2\omega$ sen ϕ is the coriolis parameter, ϕ is the latitude, ω the angular velocity of the earth about its axis, $\varkappa = C_p/C_v$, C_p and C_v the specific heat capacity at constant pressure and at constant volume, g the gravitational acceleration. Rotation is included in order to generalize although it is irrelevant to the main issue.

These equations can be linearized and using fourier series expansions for the disturbances in the fields of the pressure, density and velocity, it is easy to arrive at the following equation (see for example Musaelyan, 1964).

$$\frac{d^2\varphi_{ij}}{dz^2} + \left(-\tfrac{1}{4}\sigma^2 - \tfrac{1}{2}\frac{d\sigma}{dz} + D_{ij}\right)\varphi_{ij} = 0 \qquad (4)$$

where φ_{ij} is a stream function depending on z only, and σ and D_{ij} coefficients given by the following relations:

$$\left.\begin{aligned} \sigma(z) &= \frac{\gamma_a - \gamma}{T} + \frac{g}{\varkappa RT} \\ D_{ij}(z) &= g\frac{k^2 + m^2}{\bar{u}^2 k^2 - f^2}\frac{\gamma_a - \gamma}{T} \end{aligned}\right\} \qquad (5)$$

Taking into account the order of magnitude of the terms involved, (4) can be simplified and we have the final equation

$$\frac{d^2\varphi_{ij}}{dz^2} + g\frac{k^2 + m^2}{\bar{u}^2 k^2 - f^2}\frac{\gamma_a - \gamma}{T}\varphi_{ij} = 0 \qquad (6)$$

(k, m are the wave numbers) or

$$\frac{d^2\varphi_{ij}}{dz^2} + \frac{k^2 + m^2}{1 - \dfrac{f^2}{\bar{u}^2 k^2}} l^2 \varphi_{ij} = 0 \qquad (7)$$

At this point many authors introduce two hypotheses concerning the boundary conditions, (7) must satisfy:

Hypothesis 1

$$w = 0 \quad f \text{ or } z = H$$

H is for example the height of the tropopause.

Hypothesis 2

In the up-stream side of the mountain the non-disturbed flow is two-dimensional (plane X, Y) and a function only of height. At ∞

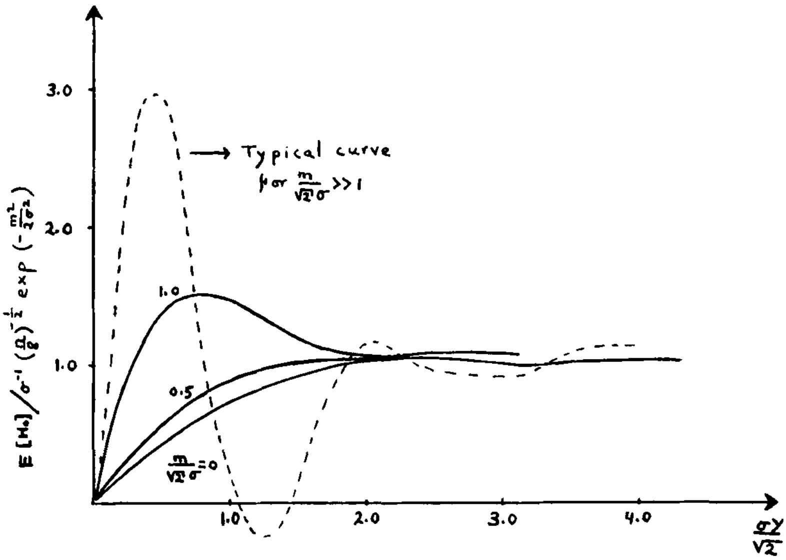

Fig. 1. Mean weight function as function of the variable y, $y = z - \xi$ (ref. 6).

$$\mathbf{u}_\infty = u_\infty(z_\infty)\,\mathbf{i}$$

the velocity far away at the up-stream side of the mountain is for $x \to -\infty$ $u = u_\infty(z_\infty)$

$$v = w = 0$$

In other words, no waves occur up-stream.

Hypothesis number one is highly simplified as it is known that disturbances produced by low hills can extend to much greater heights and may be the cause of the formation of certain stratospheric clouds.

3. Solution for a random Gaussian parameter of stability

Let us now see what happens when l^2 is taken as a random coefficient, eq. (7) can be written as

$$\frac{d^2\Psi}{dz^2} + a(z)\Psi = 0 \quad (8)$$

The only restriction upon $a(z)$ is that all derivatives of the coefficient are negligible with respect to the coefficient. Let us suppose $a(z)$ is given by a χ^2 distribution

$$\text{if } \alpha = \pm a^{\frac{1}{2}}$$

and $\quad E\{\alpha\} = m = \text{mean}$

$$E\{(\alpha - m)^2\} = \sigma^2 \quad (\sigma = \text{standard deviation}) \quad (9)$$

the probability law for α is

$$F(\alpha) = (2\pi)^{-\frac{1}{2}}\sigma^{-1} \exp\left[-(\alpha - m)^2/2\sigma^2\right] \quad (10)$$

using a method developed by Samuels and Eringen (ref. 6) this equation can be solved for the mean Green's function.

$$E\{H_0\} = \tfrac{1}{4}\sigma^{-1}(2\pi)^{\frac{1}{2}} \exp\left(-\frac{m^2}{2\sigma^2}\right) \times \left[\operatorname{erf}\left\{\frac{\sigma}{\sqrt{2}}\left(y - \frac{im}{\sigma^2}\right)\right\} + \operatorname{erf}\left\{\frac{\sigma}{\sqrt{2}}\left(y + \frac{im}{\sigma^2}\right)\right\}\right] \quad (11)$$

erf = error function.

The Green's function $H(z, \xi)$ is the solution of the equation

$$\frac{d^2H}{dz^2} + aH = \delta(z - \xi)$$

under the condition that $H = 0$ for all $z < \xi$. The solution of this equation can be assumed to be of the form

$$H(z, \xi) = H_0 + H_1(z; \xi)$$

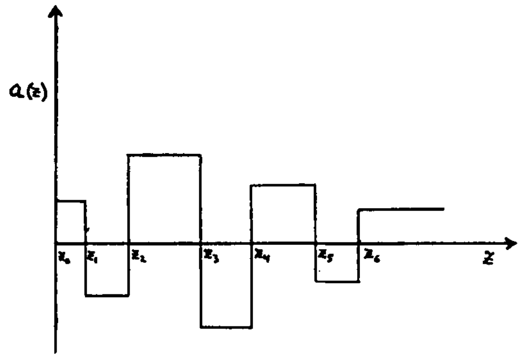

Fig. 2. Hopping Markov process, the points Z_0, Z_1, Z_2, ..., Z_n are distributed according to a Poisson law and the heights according to a Gaussian probability law.

where H_1 is a correction term.

$$y = z - \xi$$

If we plot

$$E\{H_0\} \text{ vs. height } \left(\frac{\sigma}{\sqrt{2}} y\right)$$

see Fig. 1, we can deduce the following conclusions:

1. There is a considerable damping of the waves with height, and for $m/\sqrt{2}\sigma < 1$ there are no waves at all.

2. Near the top of the mountain for $m/\sqrt{2}\sigma \gg 1$ there is a considerable amplification of the mean intensity of the basic flow.

3. In any case a steady state is reached. ($m/\sqrt{2}\sigma$ is a dimensionless number characterizing the strength of random fluctuations of the medium. The quantity generally used is σ/m.)

We see then, that in the case of a dry atmosphere, hypothesis 1 is not needed. Waves do not propagate up to a considerable height. There is only a gusty wind.

4. Analysis of the solution for a discontinuous Markov chain as random parameter of stability

The second case to be considered is where $a(z)$ can be described by a special kind of Markov process, i.e. the dichotomic Markov chain, which is also known as the random telegraph signal. Dichotomic refers to a two-valued Markov process, and is an over-simplification of the process describing the fluctuations due to condensation, of the parameter of stability. This process could be better described by the following fig. 2.

The D.M.P. is fig. 3.

We can write eq. (7) in a non-dimensional form, if we introduce the generalized Reynolds number

$$R_g = \epsilon K_0 L^2 \quad \text{(Frisch, 1968)}$$

where:

K_0 = characteristic wave number for free space

L = characteristic correlation length scale of turbulence

ε = σ/m = the relative strength of fluctuations

(7) becomes:

$$\frac{d^2\Psi(z)}{dz^2} + K_0^2 \varepsilon L^2 s(z)\, \Psi(z=0) \tag{12}$$

with

$$K_0^2 = g\frac{k^2+m^2}{\bar{u}^2k^2 - f^2}$$

and $s(z)$ as the random function. In Fig. 2, the density of the Poisson law which gives the distribution of the intervals on the z acis, can now be expressed as:

$$c = (K_0 L)^{-1} \tag{13}$$

The probability of having n transitions in an interval of length z is:

$$P_n(z) = \frac{1}{n!}(cz)^n e^{-cz} \tag{14}$$

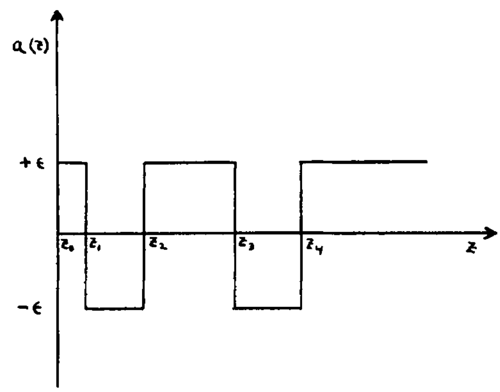

Fig. 3. Dichotomic Markov chain. Z_0, Z_1, Z_2 ..., Z_n are distributed as in Fig. 2, according to a Poisson law. Dichotomic refers to a two-valued random function with zero mean (Bourret, 1964).

between two transitions $s(z)$ keeps a constant value determined by a Gaussian probability law of the form

$$F(a) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp - \frac{t^2}{2\sigma^2} \qquad (15)$$

$$t^2 = (a-m)^2$$

The probability of having in the interval $(0, Z)$ one transition and, only one, between z_1 and $z_1 + dz_1$ is:

$$e^{-cz} c dz_1 \qquad (16)$$

If $P(s)$ is the distribution law of the values of $s(z)$ at a given height and if $P(R)$ is the probability law of having n jumps in the elementary intervals

$$(z_1, z_1 + dz_1) \ldots (z_n, z_n + dz_n)$$

with the $n+1$ values of s in the intervals

$$(s_0, s_0 + ds_0) \ldots (s_n, s_n + ds_n)$$

we have

$$P(R)\, dz_1 \ldots dz_n ds_0 \ldots ds_n$$

$$= [P(s_0)\, ds_0 e^{-c(z_1 - z_0)} c dz_1]\, x$$

$$\times \ldots [P(s_n)\, ds_n e^{-c(z - zn)}] \qquad (17)$$

In the Dichotomic Markov Process (Fig. 3), the problem is simplified and the random function can take only two values $+\varepsilon$ and $-\varepsilon$.

Eq. (12) can be expressed in vector form in the following way:

$$\frac{d}{dz}\, \mathfrak{c}(z) = A[s(z)]\, \mathfrak{c}(z)$$

$\mathfrak{c}(z)$ is the vector

$$\begin{cases} y_1(z) \\ y_2(z) \end{cases} \quad A[\quad] = \begin{pmatrix} 0 & K_0 \\ -K_0\, \varepsilon\, Ls(z) & 0 \end{pmatrix} \qquad (18)$$

Using a method developed by Bharucha-Reid (1964) and expression for the mean of $\mathfrak{c}(z)$ can be obtained

$$E[\mathfrak{c}(z)]$$

$$= \Bigg\{ R(z - z_0)\, e^{-c(z - z_0)}$$

$$+ \int_{z_0}^{z} c dz_1\, R(z - z_1)\, e^{-(z - z_1)} R(z_1 - z_0) e^{-c(z_1 - z_0)}$$

$$+ - - - - - - - - - - -$$

$$+ \int_{z_0}^{z} c\, dz_1 \int_{z_1}^{z} \ldots \int_{z n-1}^{z} c\, d_n\, R(z - z_n)\, e^{-c(z - z n)}$$

$$\times \ldots \times R(z_1 - z_0)\, e^{-c(z_1 - z_0)} + \ldots + \Bigg\}\, \mathfrak{c}(z_0) \qquad (19)$$

$R(z)$ is given by

$$R(z) = \int_{\tau} ds\, P(s)\, R_0^z(s) \qquad (20)$$

$R(z)$ is nothing but the mean of the "resolvents" of the matrix A. The resolvent of A between z_0 and z_1 is by definition:

$$R_{z_0}^{z_1}$$

$$= \left\{ [I] + \int_{z_0}^{z_1} (\;)\, dze + \int_{z_0}^{z_1} (\;)\, dz_1 \int_{z_0}^{z_1} (\;)\, dz_2 + \ldots + \right\}$$

If we restrict ourselves to the Dichotomic Markov Process, and if we take the Laplace transform of (19) we can analyse (19) without direct integration

$$\bar{\bar{\mathfrak{c}}}(z) = (\bar{\bar{R}}_0(p) + c\bar{\bar{R}}_0^2(p) + \ldots + c^{n-1} \bar{\bar{R}}_0^n(p))\, \mathfrak{c}(z_0) \qquad (21)$$

= indicates the transform of ...

p is the complex variable.

In the D.M.P. $R(z)$ has only two forms $\pm R_0$, R_0 can be easily determined from the matrix A with the aid of Baker's formulas.

(21) can be expressed in the form

$$\bar{\bar{\mathfrak{c}}}(z) = \frac{\bar{\bar{R}}_0(p)}{\bar{\bar{I}} - c\bar{\bar{R}}(p)}\, \mathfrak{c}(z_0) \qquad (22)$$

The characteristic equation of (22) is:

$$|[I] - c\bar{\bar{R}}_0(p)| = 0 \qquad (23)$$

It is easy to verify that it is of the fourth order. This equation gives us the eigensolutions of (12). Each pole $P_j = u_j + iv_j$ yields a contribution proportionnal to

$$\exp\{-iu_j z\} \exp\{-v_j z\}$$

which has wave number u_j and damping length v_j^{-1}. For the following values of the basic parameters (Queney, 1947)

$$\left.\begin{aligned} K_0 &= 1 \times 10^{-2}\, mts^{-1} \\ L &= 3 \times 10^{2}\, mts \\ \varepsilon &= 0.35 \end{aligned}\right\}$$

we find using Graeffe's method and Routh–Heurwitz stability criterion, that there is one pair of complex roots and two unstable roots.

In general, roots are located near the imaginary axis of the complex plane, so that there is weak stability and the possible existence of unstable solutions.

By unstable solutions we mean, undamped solutions, that is waves with an amplitude that increases with the height z. Queney and Lyra have obtained similar solutions for the steady motion. Usually this is explained as being a result of not taking into account the variation of static stability and wind with height (Eliassen & Kleinschmidt, 1957). We see here, that this explanation is true only when we consider Gaussian fluctuations of the stability parameter, but for Markovian fluctuations of the D.M.P. kind, the reverse is true.

W can now try, to give a physical interpretation to these undamped waves.

Palm and Wurtele (ref. 9) have treated the problem of mountain waves as an initial value problem, and shown that the stationary motion is independent of initial conditions. The existence of undamped waves for the steady state, then means that actually, no stationary motion will finally be reached outside the vicinity of the mountain, and following Von Neumann's analysis of stability (1963), the unstable modes can develop turbulence with appearance of vortexes. In this case waves can propagate to a considerable height beyond the mountain.

5. Conclusions

We may conclude then, that the development of mountain waves and associated turbulence depends strongly on the humidity of the atmosphere. A process like the one shown in Fig. 2, diminishes the stability of the basic current and eventually develops turbulence. In this case we have an example of a combined barotropic inertial and convective instability which can be seen as a sort of baroclinic instability. This result agrees with the analysis of billow mechanics made by Scorer (1969).

We think we have exposed some mathematical methods developed in recent years which can contribute something to the study of related problems.

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REFERENCES

Blinova, E. N. 1965. *Long range hydrodynamic weather forecasting*. Israel Program for Scientific Translations, Jerusalem.

Ellison, T. H. 1951. The propagation of sound waves through a medium with very small random fluctuations of refractive index. *J. Atmos. and Terr. Phys. 2*, 14–21.

Frisch, U. 1968. Wave propagation in random media. In *Probabilistic methods in applied mathematics*, pp. 75–198. Academic Press, New York.

Chen, Y. S. 1962. The response of the atmosphere to the solar actual direct radiation in a linear model of the long-range numerical prediction. *Scientia Sinica II*, 275–291.

Musaelyan, Sh. A. 1964. *Barrier waves in the atmosphere*. Israel Program for Scientific Translations, Jerusalem.

Samuels, J. & Eringen, C. 1958. On stochastic linear systems. *Journal of Math. and Physics 38*, 83–103.

Bharucha-Reid, A. T. 1964. On the theory of random equations. In *Proc. Symp. Appl. Math. 16*. Providence R.I. Amer. Math. Soc.

Queney, P. 1947. *Theory of perturbations in stratified currents with applications to air flow over mountain barriers*. The University of Chicago Press, Chicago, Ill.

Eliassen, A. & Kleinschmidt, E. 1957. Dynamic meteorology. In *Handbuch der Physik*, Band XLVIII, Geophysik II, pp. 59–64. Springer-Verlag, Berlin, Göttingen and Heidelberg.

Von Neumann, J. 1963. Recent theories of turbulence. In *Collected Works*, vol. VI, pp. 437–472. Pergamon Press, Oxford, London, New York and Paris.

Scorer, R. S. 1969. Billow mechanics. *Radio Science 4*, no. 12, pp. 1299–1308.

ГОРНЫЕ ВОЛНЫ В ТУРБУЛЕНТНОЙ АТМОСФЕРЕ

Изучается распространение волн за горами в атмосфере с флуктуирующими параметрами устойчивости. Рассматриваются два случая, гауссовый и случай дискретных марковских цепей, описывающие сухую и влажную атмосферу, соответственно.