Worked Examples from Introductory Physics (Algebra–Based)
Vol. I: Basic Mechanics

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Contents

Preface

This booklet can be downloaded free of charge from:

`http://iweb.tntech.edu/murdock/books.html`

The date on the cover page serves as an edition number. I'm continually tinkering with these booklets.

This book is:

- A summary of the material in the first semester of the non–calculus physics course as I teach it at Tennessee Tech.
- A set of example problems typical of those given in non–calculus physics courses solved and explained as well as I know how.

It is not intended as a substitute for any textbook suggested by a professor... at least not yet! It's just here to help you with the physics course you're taking. Read it alongside the text they told you to buy. The subjects should be in the *rough* order that they're covered in class, though the chapter numbers won't exactly match those in your textbook.

Feedback and errata will be appreciated. Send mail to me at:

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Chapter 1

Mathematical Concepts

1.1 The Important Stuff

1.1.1 Measurement and Units in Physics

Physics is concerned with the relations between measured quantities in the natural world. We make measurements (length, time, etc) in terms of various **standards** for these quantities.

In physics we generally use the "metric system", or more precisely, the **SI** or **MKS** system, so called because it is based on the **meter**, the **second** and the **kilogram**.

The meter is related to basic length unit of the "English" system —the inch— by the *exact* relations:

$$1\,\mathrm{cm} = 10^{-2}\,\mathrm{m} \qquad \text{and} \qquad 1\,\mathrm{in} = 2.54\,\mathrm{cm}$$

From this we can get:

$$1\,\mathrm{m} = 3.281\,\mathrm{ft} \qquad \text{and} \qquad 1\,\mathrm{km} = 0.6214\,\mathrm{mi}$$

Everyone knows the (exact) relations between the common units of time:

$$1\,\text{minute} = 60\,\text{sec} \qquad 1\,\text{hour} = 60\,\text{min} \qquad 1\,\text{day} = 24\,\text{h}$$

and we also have the (pretty accurate) relation:

$$1\,\text{year} = 365.24\,\text{days}$$

Finally, the unit of **mass** is the kilogram. The *meaning* of mass is not so clear unless you have already studies physics. For now, suffice it to say that a *mass* of 1 kilogram has a *weight* of — pounds. Later on we will make the distinction between "mass" and "weight".

1.1.2 The Metric System; Converting Units

To make the SI system more convenient we can associate prefixes with the basic units to represent powers of 10. The most commonly used prefixes are given here:

Factor	**Prefix**	**Symbol**
10^{-12}	pico-	p
10^{-9}	nano-	n
10^{-6}	micro-	μ
10^{-3}	milli-	m
10^{-2}	centi-	c
10^{3}	kilo-	k
10^{6}	mega-	M
10^{9}	giga-	G

Some examples:

$$1\,\text{ms} = 1\,\text{millisecond} = 10^{-3}\,\text{s}$$

$$1\,\mu\text{m} = 1\,\text{micrometer} = 10^{-6}\,\text{s}$$

Oftentimes in science we need to change the units in which a quantity is expressed. We might want to change a length expressed in feet to one expressed in meters, or a time expressed in days to one expressed in seconds.

First, be aware that in the math we do for physics problems a unit symbol like 'cm" (centimeter) or "yr" (year) is treated as a multiplicative factor which we can cancel if the same factor occurs in the numerator and denominator. In any case we can't simply ignore or erase a unit symbol.

With this in mind we can set up conversion factors, which contain the same *quantity* on the top and bottom (and so are equal to 1) which will cancel the old units and give new ones.

For example, 60 seconds is equal to one minute. Then we have

$$\left(\frac{60\,\text{s}}{1\,\text{min}}\right) = 1$$

so we can multiply by this factor without changing the *value* of a number. But it can give us new *units* for the number. To convert 8.44 min to seconds, use this factor and cancel the symbol "min":

$$8.44\,\text{min} \;=\; (8.44\,\text{min})\left(\frac{60\,\text{s}}{1\,\text{min}}\right) \;=\; 506\,\text{s}$$

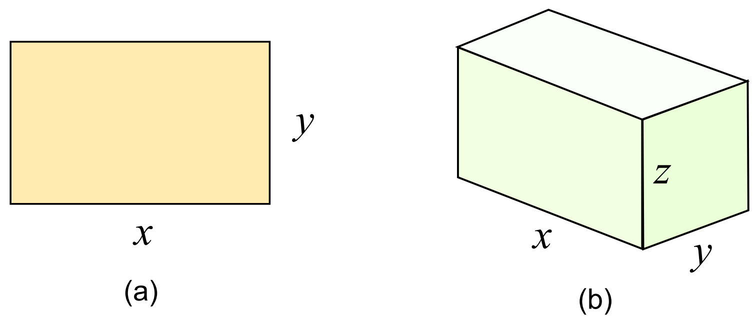

Figure 1.1: (a) Rectangle with sides x and y. Area is $A = xy$. I hope you knew that. (b) Rectangular box with sides x, y and z. Volume is $V = xyz$. I hope you knew that too.

If we have to convert 3.68×10^4 s to minutes, we would use a conversion factor with seconds in the denominator (to cancel what we've got already; the conversion factor is still equal to 1). So:

$$3.68 \times 10^4\,\mathrm{s} \;=\; (3.68 \times 10^4\,\mathrm{s}) \left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) = 613\,\mathrm{min}$$

1.1.3 Math: You Had This In High School. Oh, Yes You Did.

The mathematical demands of a "non–calculus" physics course are not extensive, but you do have to be proficient with the little bit of mathematics that we *will* use! It's just the stuff you had in high school. Oh, yes you did. Don't tell me you didn't.

We will often use scientific notation to express our numbers, because this allows us to express large and small numbers conveniently (and also express the precision of those numbers). We will need the basic algebra operations of powers and roots and we will solve equations to find the "unknowns".

Usually the algebra will be very simple. But if we are ever faced with an equation that looks like

$$ax^2 + bx + c = 0 \tag{1.1}$$

where x is the unknown and a, b and c are given numbers (constants) then there are two possible answers for x which you can find from the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.2}$$

On occasion you will need to know some facts from geometry. Starting simple and working upwards, the simplest shapes are the rectangle and rectangular box, shown in Fig. 1.1. If

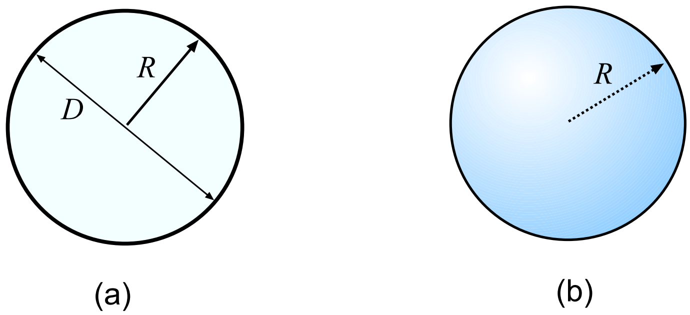

Figure 1.2: (a) Circle; $C = \pi D = 2\pi R$; $A = \pi R^2$. (b) Sphere; $A = 4\pi R^2$; $V = \frac{4}{3}\pi R^3$. You've seen these formulae before. Oh, yes you have.

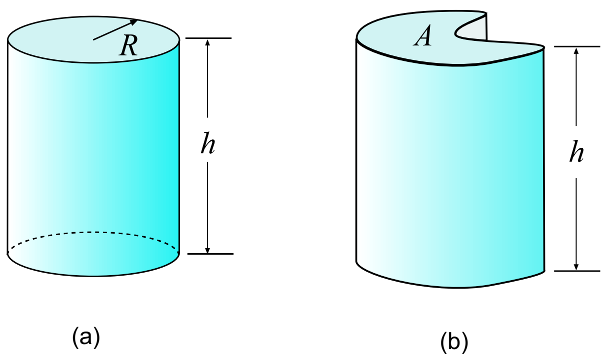

Figure 1.3: (a) Circular cylinder of radius R and height h. Volume is $V = \pi R^2 h$. (b) Right cylinder of arbitrary shape. If the area of the cross section is A, the volume is $V = Ah$.

the rectangle has sides x and y its area is $A = xy$. Since it is the product of two *lengths*, the units of area in the SI system are m^2. For the rectangular box with sides x, y and z, the volume is $V = xyz$. A volume is the product of *three* lengths so its units are m^3.

Other formulae worth mentioning here are for the circle and the sphere; see Fig. 1.2.

A circle is specified by its radius R (or its diameter D, which is twice the radius). The distance around the circle is the circumference, C. The circumference and area A of the circle are given by

$$C = \pi D = 2\pi R \qquad\qquad A = \pi R^2 \tag{1.3}$$

A sphere is specified by its radius R. The surface area A and volume V of a sphere are given by

$$A = 4\pi R^2 \qquad\qquad V = \tfrac{4}{3}\pi R^3 \tag{1.4}$$

Another simple shape is the (right) circular cylinder, shown in Fig. 1.3(a). If the cylinder has radius R and height h, its volume is $V = \pi R^2 h$. This is a special case of the general right cylinder (see Fig. 1.3(b)) where if the area of the cross section is A and the height is h, the volume is $V = Ah$.

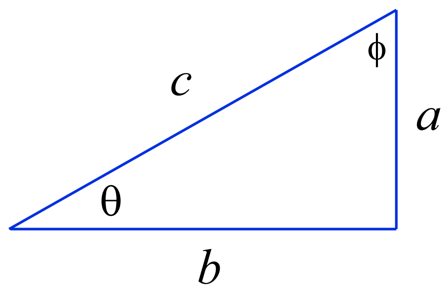

Figure 1.4: Right triangle with sides a, b and c.

1.1.4 Math: Trigonometry

You will also need some *simple* trigonometry. This won't amount to much more than relating the sides of a **right triangle**, that is, a triangle with two sides joined at 90°.

Such a triangle is shown in Fig. 1.4. The sides a, b and c are related by the **Pythagorean Theorem**:

$$a^2 + b^2 = c^2 \qquad \Longrightarrow \qquad c = \sqrt{a^2 + b^2} \tag{1.5}$$

We only need the angle θ to determine the *shape* of the triangle and this gives the *ratios* of the sides of the triangle. The ratios are given by:

$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c} \qquad \tan\theta = \frac{a}{b} \tag{1.6}$$

Or you can remember these ratios in term of their positions with respect to the angle θ. If the sides are

$$a = \text{opposite} \qquad b = \text{adjacent} \qquad c = \text{hypothenuse}$$

then the ratios are

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} \tag{1.7}$$

If you pick out the first letters of the "words" in Eq. 1.7 in order, they spell out *SOHCAH-TOA*. If you want to remember the trig ratios by intoning "$\mathcal{SOHCAHTOA}$", be my guest, but don't do it near me.

1.1.5 Vectors and Vector Addition

Throughout our study of physics we will discuss quantities which have a *size* (that is, a **magnitude**) as well as a *direction* These quantities are called **vectors**. Examples of vectors are velocity, acceleration, force, and the electric field.

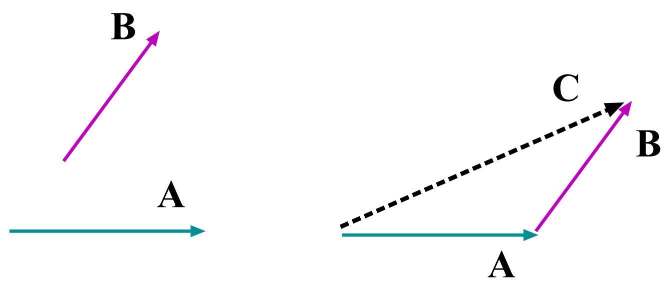

Figure 1.5: Vectors **A** and **B** are added to give the vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

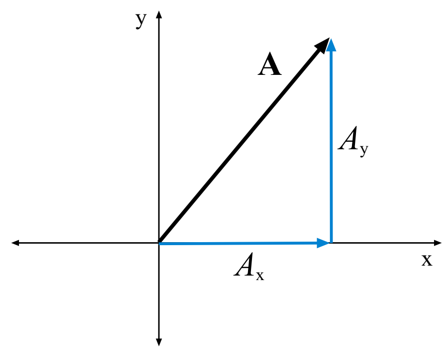

Figure 1.6: Vector **A** is split up into components.

Vectors are represented by arrows which show their magnitude and direction. The laws of physics will require us to *add* vectors, and to represent this operation on paper, we *add the arrows*. The way to add arrows, say to add arrow **A** to arrow **B** we join the tail of **B** to the head of **A** and then draw a new arrow from the tail of **A** to the head of **B**. The result is $\mathbf{A} + \mathbf{B}$. This is shown in Figure 1.5.

Vectors can be multiplied by ordinary numbers (called **scalars**), giving new vectors, as shown in Fig. 1.5.

1.1.6 Components of Vectors

Addition of vectors would be rather messy if we didn't have an easy technique for handling the trigonometry. Vector addition is made much easier when we split the vectors into parts that run along the x axis and parts that run along the y axis. These are called the x and y **components** of the vector.

In Figure 1.6A vector split up into components: One component is a vector that runs along the x axis; the other is one running along the y axis.

If we let A be the magnitude of vector **A** and θ is its direction as measured counter–

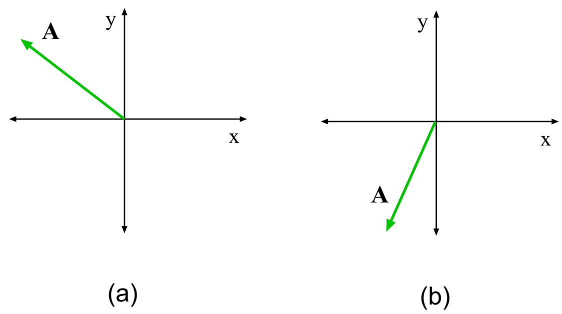

Figure 1.7: Vectors can have negative components when they're in the other quadrants.

clockwise from the $+x$ axis, then the component of this vector that runs along x has length A_x, where the relation between the two is:

$$A_x = A\cos\theta \tag{1.8}$$

Likewise, the length of the component that runs along y is

$$A_y = A\sin\theta \tag{1.9}$$

Actually, we don't literally mean "length" here since that implies a positive number. When the vector **A** has a direction lying in quadrants II, III or IV (as in Figure 1.7, then one of its components will be negative. For example, if the vector's direction is in quadrant II as in Fig. 1.7(a), its x component is negative while its y component is positive.

Now if we have the components of a vector we can find its magnitude and direction by the following relations:

$$A = \sqrt{A_x^2 + A_y^2} \qquad \tan\theta = \frac{A_y}{A_x} \tag{1.10}$$

where θ is the angle which gives the direction of **A**, measured counterclockwise from the $+x$ axis.

Once we have the x and y components of two vectors it is easy to add the vectors since the x components of the individual vectors add to give the x component of the sum, and the y components of the individual vectors add to give the y component of the sum. This is illustrated in Figure 1.8. Expressing this with math, if we say that $\mathbf{A} + \mathbf{B} = \mathbf{C}$, we mean

$$A_x + B_x = C_x \qquad \text{and} \qquad A_y + B_y = C_y \tag{1.11}$$

One we have the x and y components of the total vector **C**, we can get the magnitude and direction of **C** with

$$C = \sqrt{C_x^2 + C_y^2} \qquad \text{and} \qquad \tan\theta_C = \frac{C_y}{C_x}$$

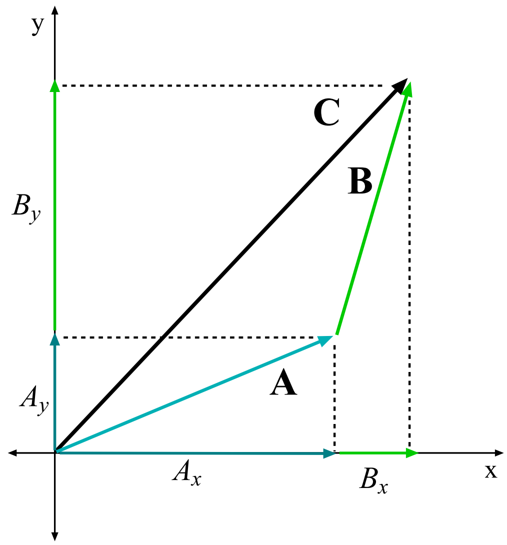

Figure 1.8: Vectors **A** and **B** add to give the vector **C**. The x components of **A** and **B** add to give the x component of **C**: $A_x + B_x = C_x$. Likewise for the y components.

Summing up, many problems involving vectors will give you the magnitudes and directions of two vectors and ask you to find the magnitude and direction of their sum. To do this,

- Find the x and y components of the two vectors.
- Add the x and y parts *individually* to get the x and y parts of the sum (resultant vector).
- Use Eq. 1.10 (trig) to get the magnitude and direction of the resultant.

1.2 Worked Examples

1.2.1 Measurement and Units

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1. The mass of the parasitic wasp *Caraphractus cintus* **can be as small as** 5×10^{-6} **kg. What is this mass in (a) grams (g), (b) milligrams (mg) and (c) micrograms (μg)?**
[CJ6 1-1]

(a) Using the fact that a kilogram is a thousand grams: $1\,\mathrm{kg} = 10^3\,\mathrm{g}$, we find

$$m = 5 \times 10^{-6}\,\mathrm{kg} = (5 \times 10^{-6}\,\mathrm{kg})\left(\frac{10^3\,\mathrm{g}}{1\,\mathrm{kg}}\right) = 5 \times 10^{-3}\,\mathrm{g}$$

(b) Using the fact that a milligram is a thousandth of a gram: $1\,\mathrm{mg} = 10^{-3}\,\mathrm{g}$, and our answer from (a), we find

$$m = 5 \times 10^{-3}\,\mathrm{g} = (5 \times 10^{-3}\,\mathrm{g}) \left(\frac{1\,\mathrm{mg}}{10^{-3}\,\mathrm{g}} \right) = 5\,\mathrm{mg}$$

(c) Using the fact that a microgram is 10^{-6} (one millionth) of a gram: $1\,\mu\mathrm{g} = 10^{-6}\,\mathrm{g}$

$$m = 5 \times 10^{-3}\,\mathrm{g} = (5 \times 10^{-3}\,\mathrm{g}) \left(\frac{1\,\mu\mathrm{g}}{10^{-6}\,\mathrm{g}} \right) = 5 \times 10^{3}\,\mu\mathrm{g}$$

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2. Vesna Vulovic survived the longest fall on record without a parachute when her plane exploded and she fell 6 miles, 551 yards**. What is the distance in meters?** [CJ6 1-2]

Convert the two lengths (i.e. 6 miles and 551 yards) to meters and then find the sum. Use the fact that 1 mile equals 1.6093 km to get:

$$6\,\mathrm{mile} = (6\,\mathrm{mile}) \left(\frac{1.6093\,\mathrm{km}}{1\,\mathrm{mile}} \right) \left(\frac{10^{3}\,\mathrm{m}}{1\,\mathrm{km}} \right) = 9656.1\,\mathrm{m}$$

and we can use the exact relation $1\,\mathrm{in} = 2.54\,\mathrm{cm}$ to get

$$\begin{aligned} 551\,\mathrm{yd} &= (551\,\mathrm{yd}) \left(\frac{36\,\mathrm{in}}{1\,\mathrm{yd}} \right) \left(\frac{2.54\,\mathrm{cm}}{1\,\mathrm{in}} \right) \left(\frac{1\,\mathrm{m}}{10^{2}\,\mathrm{cm}} \right) \\ &= 503.8\,\mathrm{m} \end{aligned}$$

Add the two lengths:

$$L_{\mathrm{Total}} = 9656.1\,\mathrm{m} + 503.8\,\mathrm{m} = 1.0160 \times 10^{4}\,\mathrm{m}$$

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3. How many seconds are there in (a) one hour and thirty–five minutes and (b) one day? [CJ6 1-3]

(a) Change one hour to seconds using the unit–factor method:

$$1\,\mathrm{h} = (1\,\mathrm{h}) \left(\frac{60\,\mathrm{min}}{1\,\mathrm{h}} \right) \left(\frac{60\,\mathrm{s}}{1\,\mathrm{min}} \right) = 3600\,\mathrm{s}$$

Likewise change 35 min to seconds:

$$35\,\text{min} = (35\,\text{min})\left(\frac{60\,\text{s}}{1\,\text{min}}\right) = 2100\,\text{s}$$

The total is

$$1\,\text{h} + 35\,\text{min} = 3600\,\text{s} + 2100s = 5700\,\text{s}$$

(b) Change one day to seconds; use the unit factors:

$$\begin{aligned} 1\,\text{day} &= (1\,\text{day})\left(\frac{24\,\text{h}}{1\,\text{day}}\right)\left(\frac{60\,\text{min}}{1\,\text{h}}\right)\left(\frac{60\,\text{s}}{1\,\text{min}}\right) \\ &= 86,400\,\text{s} \end{aligned}$$

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4. Bicyclists in the Tour de France reach speeds of 34.0 miles per hour (mi/h) on flat sections of the road. What is this speed in (a) kilometers per hour (km/h) and (b) meters per second (m/s)? [CJ6 1-4]

(a) Use the relation between miles and kilometers:

$$1\,\text{mi} = 1.609\,\text{km}$$

to get

$$v = 34.0\,\tfrac{\text{mi}}{\text{h}} = (34.0\,\tfrac{\text{mi}}{\text{h}})\left(\frac{1.609\,\text{km}}{1\,\text{mi}}\right) = 54.7\,\tfrac{\text{km}}{\text{h}}$$

(b) Using our answer from (a) along with the relations

$$1\,\text{km} = 10^3\,\text{m} \qquad \text{and} \qquad 1\,\text{hr} = (60\,\text{min})\left(\frac{60\,\text{s}}{1\,\text{min}}\right) = 3600\,\text{s}$$

to get

$$v = (54.7\,\tfrac{\text{km}}{\text{h}})\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)\left(\frac{10^3\,\text{m}}{1\,\text{km}}\right) = 15.2\,\tfrac{\text{m}}{\text{s}}$$

1.2.2 Trigonometry

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5. For the right triangle with sides as shown in Figure 1.9, find side x and the angle θ.

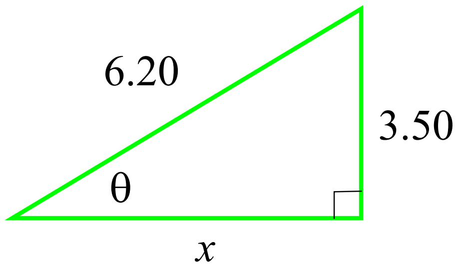

Figure 1.9: Right triangle for example 5.

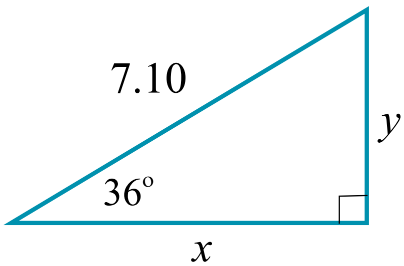

Figure 1.10: Right triangle for example 6.

We can use the Pythagorean theorem to find x. Pythagoras tells us:

$$x^2 + (3.50)^2 = (6.20)^2$$

Solving for x gives

$$x^2 = (6.20)^2 - (3.50)^2 = 26.19 \qquad \Longrightarrow \qquad x = \sqrt{26.19} = 5.12$$

As for θ, since we are given the "opposite" side and the hypothenuse, we know $\sin\theta$. It is:

$$\sin\theta = \frac{3.50}{6.20} = 0.565$$

Then get θ with the inverse sine operation:

$$\theta = \sin^{-1}(0.565) = 34.4^\circ$$

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6. For the right triangle with the side and angle as shown in Figure 1.10, find the missing sides x and y.

We don't know the "opposite" side y but we do know the angle to which it is opposite. So we can write a relation involving the sine of the angle, thus:

$$\sin 36^\circ = \frac{y}{7.10} \qquad \Longrightarrow \qquad y = (7.10)\sin 36^\circ = 4.17$$

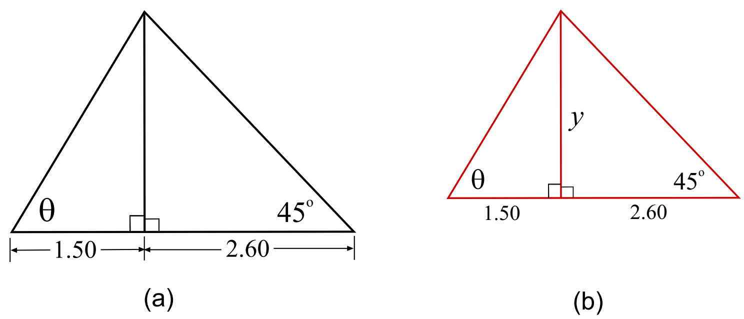

Figure 1.11: Right triangle for example 7.

Likewise, we can write a relation involving the "adjacent" side and the cosine of the angle,

$$\cos 36^\circ = \frac{x}{7.10} \qquad \Longrightarrow \qquad x = (7.10)\cos 36^\circ = 5.74$$

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7. Find the missing angle θ in Figure 1.11(a). (The right angles in the figure are marked.)

It will help to first find the length of the side marked y in Fig. 1.11(b). Since y and the side of length 2.60 are the opposite and adjacent sides of the 45° angle, we have:

$$\tan 45^\circ = \frac{y}{(2.60)} \qquad \Longrightarrow \qquad y = (2.60)\tan 45^\circ = 2.60$$

We can write a similar relation for the missing angle,

$$\tan\theta = \frac{y}{(1.50)} = \frac{2.60}{1.50} = 1.73$$

Using the inverse tangent operation,

$$\theta = \tan^{-1}(1.73) = 60.0^\circ$$

The missing angle is 60.0°.

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8. You are driving into St. Louis, Missouri and in the distance you see the famous Gateway–to–the–West arch. This monument rises to a height of 192 m. You estimate your line of sight with the top of the arch to be 2.0° above the horizontal. Approximately how far (in kilometers) are you from the base of the arch? [CJ6 1-11]

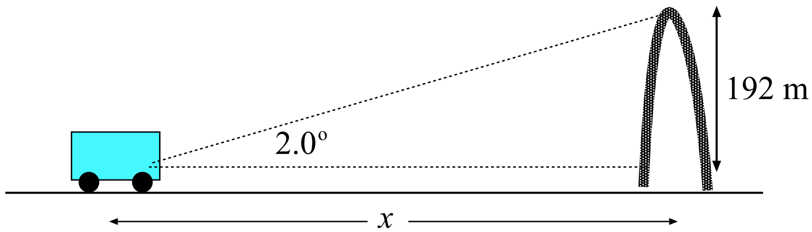

Figure 1.12: Gateway Arch is viewed from car.

The situation is diagrammed in Figure 1.12. (Of course the ground is not exactly flat and your eyeballs are not quite at ground level but these details don't make much difference.)

If the distance of the car from the base of the arch is x then we have

$$\frac{(192\,\mathrm{m})}{x} = \tan(2.0^\circ) = 3.49 \times 10^{-2}$$

Solve for x:

$$\begin{aligned} x &= \frac{(192\,\mathrm{m})}{(3.49 \times 10^{-2})} = 5.50 \times 10^3\,\mathrm{m} \\ &= 5.50\,\mathrm{km} \end{aligned}$$

The car is about 5.50 km from the base of the arch.

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9. The silhouette of a Christmas tree is an isosceles triangle. The angle at the top of the triangle is 30.0°**, and the base measures** $2.00\,\mathrm{m}$ **across. How tall is the tree?** [CJ6 1-15]

The triangle described in the problem is shown in Fig. 1.13(a). By "isosceles" we mean that the two angles at the bottom are the *same* and as a result the two sides have the same length.

We can drop a line from the top of the triangle to the base; this line divides the base into two equal parts, and since the length of the whole base is 2.0 m, the length of each part is 1.0 m. This is shown is Fig. 1.13(b). Let the height of the triangle be called y.

Now since the angles in a triangle must all add up to 180° we have

$$2\theta + 30^\circ = 180^\circ \qquad \Longrightarrow \qquad 2\theta = 150^\circ \qquad \Longrightarrow \qquad \theta = 75^\circ$$

and then we can write

$$\tan\theta = \frac{y}{1.00\,\mathrm{m}}$$

and then solve for y:

$$y = (1.00\,\mathrm{m})\tan\theta = (1.00\,\mathrm{m})\tan 75^\circ = 3.73\,\mathrm{m}$$

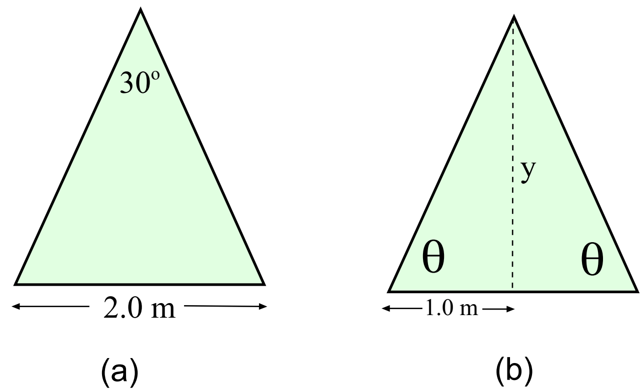

Figure 1.13: Isoceles–triangle shaped Christmas tree

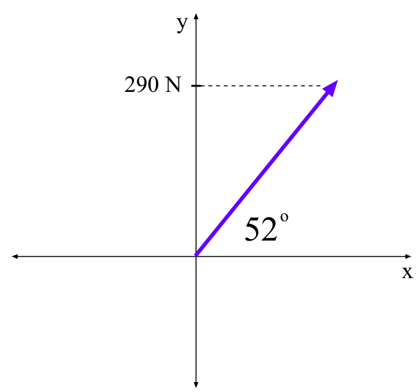

Figure 1.14: Force vector for Example 10.

1.2.3 Vectors and Vector Addition

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10. A force vector points at an angle of 52° above the $+x$ axis. It has a y component of $+290\,\mathrm{newtons}$. Find (a) the magnitude and (b) the x component of the force vector. [CJ6 1-38]

(a) The vector (which we'll call **F**) is shown in Fig. 1.14. We know F_y and the direction of **F**. With F standing for the magnitude of F, we have

$$\sin(52^\circ) = \frac{F_y}{F} = \frac{(290\,\mathrm{N})}{F}$$

Then solve for F:

$$F = \frac{(290\,\mathrm{F})}{(\sin 52^\circ)} = 368\,\mathrm{N}$$

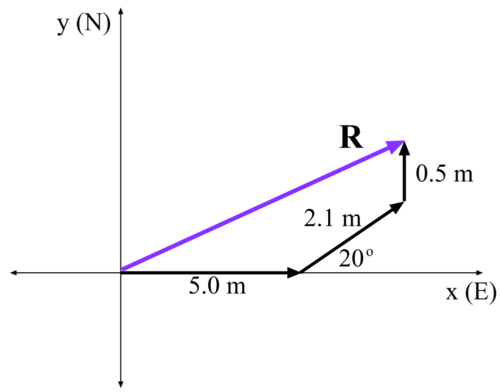

Figure 1.15: Displacements of the golf ball in Example 11.

(b) We also have

$$\tan(52^\circ) = \frac{F_y}{F_x} = \frac{(290\,\mathrm{N})}{F_x}$$

Then solve for F_x:

$$F_x = \frac{(290\,\mathrm{N})}{(\tan 52^\circ)} = 227\,\mathrm{N}$$

---

11. A golfer, putting on a green, requires three strokes to "hole the ball". During the first putt, the ball rolls $5.0\,\mathrm{m}$ due east. For the second putt, the ball travels $2.1\,\mathrm{m}$ at an angle of 20.0° north of east. The third putt is $0.50\,\mathrm{m}$ due north. What displacement (magnitude and direction relative to due east) would have been needed to "hole the ball" on the very first putt? [CJ6 1-41]

The directions and magnitudes of the individual putts are shown in Fig. 1.15. The vectors are joined head–to–tail, showing the total displacement of the ball. The total displacement (which we call **R**) is also shown.

Note, the first vector only has an x component. The last vector only has a y component. We add up the $x$components of the three vectors:

$$R_x \;=\; 5.0\,\mathrm{m} + (2.1\,\mathrm{m})\cos 20^\circ + 0.0\,\mathrm{m} \;=\; 6.97\,\mathrm{m}$$

And we add up the y components of the three vectors:

$$R_y \;=\; 0.0\,\mathrm{m} + (2.1\,\mathrm{m})\sin 20^\circ + 0.50\,\mathrm{m} \;=\; 1.22\,\mathrm{m}$$

The magnitude of the net displacement is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(6.97\,\mathrm{m})^2 + (1.22\,\mathrm{m})^2} \;=\; 7.1\,\mathrm{m}$$

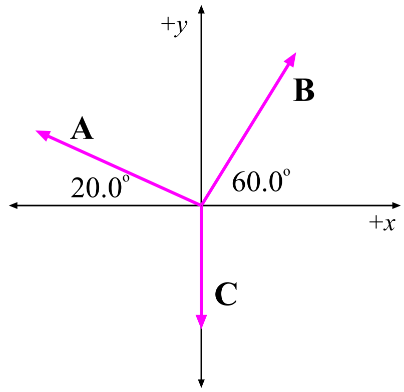

Figure 1.16: Vectors for Example 12.

and the direction of the net displacement, as measured in the usual way ("North of East") is given by θ, where

$$\tan\theta = \frac{R_y}{R_x} = \frac{(1.22)}{(6.97)} = 0.175$$

so that

$$\theta = \tan^{-1}(0.175) = 9.9^\circ$$

Had the golfer hit the ball giving it *this* magnitude and direction, the ball would have gone in the hole with one hit, which is called a double–Bogart or something to that effect.

---

12. Find the resultant of the three displacement vectors in Fig. 1.16by means of the component method. The magnitudes of the vectors are $A = 5.00\,\mathrm{m}$, $B = 5.00\,\mathrm{m}$ and $C = 4.00\,\mathrm{m}$. [CJ6 1-43]

First find the individual components of each of the vectors. Note, the angles given in the figure are measured in different ways so we have to think about the signs of the components. Here, the x component of vector **A** is negative and the y component of vector **C** (which is all it's got!) is also negative.

Using a little trig, the components of the vectors are:

$$\begin{aligned} A_x &= -(5.00\,\mathrm{m})\cos(20.0^\circ) = -4.698\,\mathrm{m} \\ A_y &= +(5.00\,\mathrm{m})\sin(20.0^\circ) = +1.710\,\mathrm{m} \end{aligned}$$

$$\begin{aligned} B_x &= +(5.00\,\mathrm{m})\cos(60.0^\circ) = +2.500\,\mathrm{m} \\ B_y &= +(5.00\,\mathrm{m})\sin(60.0^\circ) = +4.330\,\mathrm{m} \end{aligned}$$

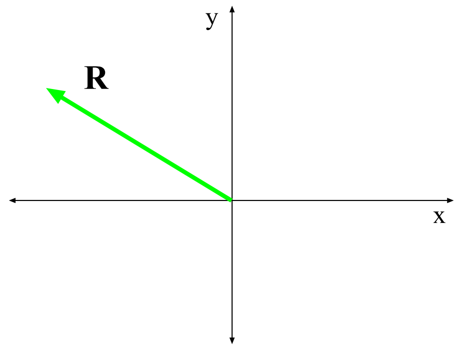

Figure 1.17: Vector **R** lies in quadrant II.

and

$$C_x = 0 \qquad C_y = -4.00\,\mathrm{m}$$

The resultant (sum) of all three vectors (which we call **R**) then has components

$$\begin{aligned} R_x &= A_x + B_x + C_x = -4.698\,\mathrm{m} + 2.500\,\mathrm{m} + 0\,\mathrm{m} = -2.198\,\mathrm{m} \\ R_y &= A_y + B_y + C_y = +1.710\,\mathrm{m} + 4.330\,\mathrm{m} - 4.000\,\mathrm{m} = 2.040\,\mathrm{m} \end{aligned}$$

This gives the *components* of **R**. The magnitude of **R** is

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2} = \sqrt{(-2.198\,\mathrm{m})^2 + (2.040\,\mathrm{m})^2} \\ &= 3.00\,\mathrm{m} \end{aligned}$$

If the direction of **R** (as measured from the $+x$ axis) is θ, then

$$\tan\theta = \frac{2.040}{(-2.198)} = -0.928$$

and naively pushing the $\tan^{-1}$ key on the calculator would have you believe that $\theta = -42.9^\circ$. Such vector would lie in the "fourth quadrant" as we usually call it. But we have found that the x component of **R** is negative while the y component is positive and such a vector must lie in the "second quadrant", as shown in Fig. 1.17. What has happened is that the calculator returns an angle that is wrong by 180° so we need to add 180° to the naive angle to get the *correct* angle. So the direction of **R** is really given by

$$\theta = -42.9^\circ + 180^\circ = 137.1^\circ$$

---

13. Vector A has a magnitude of 6.00 units and points due east. Vector B points due north. (a) What is the magnitude of B, if the vector A + B points 60.0° north of east? (b) Find the magnitude of A + B. [CJ6 1-47]

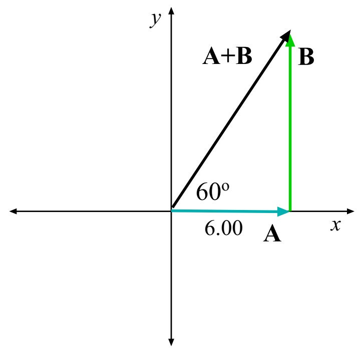

Figure 1.18: Vectors **A** and **B** for example 13.

(a) Vectors **A** and **B** are shown in Fig. 1.18. The components of **A** are

$$A_x = 6.00 \qquad A_y = 0$$

and we also know that $B_x = 0$, but we don't know B_y. But if the sum of **A** and **B** is **R**:

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

Then the components of **R** are given by

$$R_x = A_x + B + x = 6.00 + 0.00 = 6.00 \qquad R_y = A_y + B_y = 0 + B_y = B_y$$

But we are given the direction of **R**, namely $\theta = 60.0°$, so that

$$\frac{R_y}{R_x} = \tan\theta = \tan(60.0°) = 1.732$$

But then this tells us:

$$\frac{R_y}{R_x} = \frac{B_y}{A_x} = \frac{B_y}{6.00} = 1.732$$

Solve for B_y:

$$B_y = (6.00)(1.732) = 10.32$$

(b) The magnitude of $\mathbf{A} + \mathbf{B}$ (that is. **R**) is

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2} = \sqrt{(6.00)^2 + (10.32)^2} \\ &= 11.94 \end{aligned}$$

Chapter 2

Motion in One Dimension

2.1 The Important Stuff

2.1.1 Displacement

We begin with motion that takes place along a straight line, for example a car speeding up along a straight road or a rock which is thrown straight up into the air. The concepts introduced here will be useful when we solve harder problems with motion in two dimensions.

We often talk about the motion of a "particle". This just means that the object in question is small in size compared to the distance that it moves for the times of interest, so that we don't need to worry about its actual size or orientation.

We map out the possible positions of the particle with a **coordinate (system)** which might be labelled x (or y). Changes in position are given by changes in the value of x; we write a change in x as Δx.

The change in coordinate Δx is the **displacement** that the particle undergoes; it will occur over some time interval Δt. Displacements have units of length (meters) and can be positive or negative!

If we divide the displacement by the time interval we get the **average velocity** for the particle for the given time period Δt.

2.1.2 Speed and Velocity

When an object undergoes a displacement Δx in a time interval Δt, the ratio is the **average velocity** $\overline{v}$ for that time interval:

$$\overline{v} = \frac{\Delta x}{\Delta t} \tag{2.1}$$

Velocity has units of length divided by time; in physics, we will usually express velocity in $\frac{\text{m}}{\text{s}}$.

The average velocity depends on the time interval chosen for the measurement Δx and as such isn't a very useful quantity as far as physics is concerned. A more useful idea is that of a velocity associated with a *given moment in time*. This is found by calculating $\overline{v}$ for a *very small* time interval Δt which includes the time t at which we want this velocity.

The **instantaneous velocity** v is given by:

$$v = \frac{\Delta x}{\Delta t} \qquad \text{for "very small" } \Delta t. \tag{2.2}$$

The instantaneous velocity has a *definite value* at *each point in time*.

The idea of an *instantaneous* velocity is familiar from the fact that you can tell the speed of a car *at a given time* by looking at its speedometer. Your speedometer might tell you that you are travelling at $65\,\frac{\text{mi}}{\text{hr}}$. That doesn't mean that you intend to drive 65 mi or that you intend to drive for 1 hour! It means what Eq. 2.2 says: At the time you looked at the speedometer, a small displacement of the car divided by the corresponding small time interval gives $65\,\frac{\text{mi}}{\text{hr}}$. (Of course, when we use the idea in physics, we use the metric system! We will us $\frac{\text{m}}{\text{s}}$.)

The concept of taking a ratio of terms which are "very small" is central to the kind of mathematics known as **calculus**. Even though this course is supposed to be "non–calculus" we have to cheat a little because the idea of instantaneous velocity is so important!!

2.1.3 Motion With Constant Velocity

When an object starts off at the origin (so that $x = 0$ at time $t = 0$) and its velocity is *constant*, then

$$x = v_0 t \qquad \text{Constant velocity!!} \tag{2.3}$$

Which is the familiar equation often stated as "distance equals speed times time". It is *only* true when the velocity of the object is constant. But in physics the really interesting cases are when the velocity is *not* constant.

2.1.4 Acceleration

We need one more idea about motion to do physics. The (instantaneous) velocity of an object can *change*. It can change slowly (as when a car gradually gets up to a cruising speed) or it can change rapidly (as when you really hit the gas pedal or the brakes in your car). The rate at which velocity changes is important in physics.

If the velocity of an object undergoes a change Δv over a time period Δt we define the **average acceleration** over that period as:

$$\overline{a} = \frac{\Delta v}{\Delta t} \tag{2.4}$$

Acceleration has units of $\frac{\text{m}}{\text{s}}$ divided by seconds (s) which we write as $\frac{\text{m}}{\text{s}^2}$.

As with velocity, the *average* quantity is not as important as the "*right–now*" quantity so we need the idea of an *instantaneous* acceleration. Therefore at any given time want to know that ratio of Δv to Δt for a *very small* change in time. The **instantaneous acceleration** a is given by:

$$a = \frac{\Delta v}{\Delta t} \qquad \text{for "very small" } \Delta t. \tag{2.5}$$

Generally the acceleration of an object can change with time. Now, since it's a free country we could ask how rapidly the *acceleration* is changing, but it turns out that this is not so important for physics. Furthermore for a great many of our problems the moving object will have a *constant* acceleration.

2.1.5 Motion Where the Acceleration is Constant

As we will see later on, the case of *constant* acceleration is encountered often because this is what happens when there is a constant *force* acting on the object. In the following equations we assume that we're talking about a particle whose acceleration a is constant.

If the object accelerates uniformly (i.e. it moves with constant a) then its velocity changes by the same amount for equal changes in the time t. We can express this as:

$$a = \frac{\Delta v}{\Delta t}$$

We will now introduce some notation that will be used in the next couple chapters: We will say that when we discuss the motion of a particle over a certain time period, the clock starts at $t = 0$. So if we ask about the velocity and position at a later time, that later time is just called "t". We will say that the velocity of the particle at $t = 0$ is v_0, and its velocity at the time t is v. Then we have $\Delta v = v - v_0$ and $\Delta t = t$, and the last equation can be rewritten as:

$$v = v_0 + at \tag{2.6}$$

Next, we ask about the displacement of the particle at time t, given that it started off with a velocity v_0. Recall that we had a formula for x in Eq. 2.3 but when there is an acceleration *that equation is no longer true!!!* (In fact it is no longer meaningful since it is not clear what "v" means.)

Again we will say that the particle is initially located at $x = 0$, that is, it is initially at the origin. Then the displacement of the particle at time t is given by:

$$x = v_0 t + \tfrac{1}{2}at^2 \tag{2.7}$$

By combining these equations we can show:

$$v^2 = v_0^2 + 2ax \tag{2.8}$$

which can be useful because it does not contain the time t. We can also show:

$$x = \tfrac{1}{2}(v_0 + v)t \tag{2.9}$$

which can be useful because it does not contain the acceleration a. But in order to *use* this equation we must know beforehand that the acceleration is *constant*.

2.1.6 Free-Fall

The most common kind of acceleration which we encounter in daily life is the one which an object undergoes when we drop it or throw it up in the air. Before stating the value of this acceleration we need to be clear about the coordinates used to describe the motion of an object in (one–dimensional) **free–fall**.

In our free–fall problems we will always have the y axis point *straight up* regardless of the initial motion of the object. So when y increases the object is moving upward and the velocity v will be *positive*; when y decreases the object is moving downward and the velocity v will be *negative*

It turns out —for reasons we can understand only after learning about forces— that when an object is moving vertically in free–fall its velocity decreases by $9.80\,\frac{\mathrm{m}}{\mathrm{s}}$ *every second*. This is true when the object is moving upward and when it is moving downward and for that matter when the object has reached its maximum height. Then the rate of change of the object's velocity has a *constant* value given by

$$a = \frac{\Delta v}{\Delta t} = \frac{(-9.80\,\frac{\mathrm{m}}{\mathrm{s}})}{(1\,\mathrm{s})} = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The minus sign is important and comes from the fact that our y axis points *upward* but things fall *downward*. This number is known as the **acceleration of gravity**.

Before going too far we should say that the acceleration of falling objects has this value over the surface of the Earth and that the value may be slightly different depending on location, i.e. at some place on earth the value may be more like $-9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

The magnitude of acceleration of gravity is such an important number in physics that we give it the name, g, so that to a good approximation we can use

$$g = 9.80\,\frac{\mathrm{m}}{\mathrm{s}^2} \tag{2.10}$$

But be careful: g is defined as a *positive* number, and with our y axis going upward, the value of a (the acceleration for a freely-falling object) is $a = -g$. Signs are important!

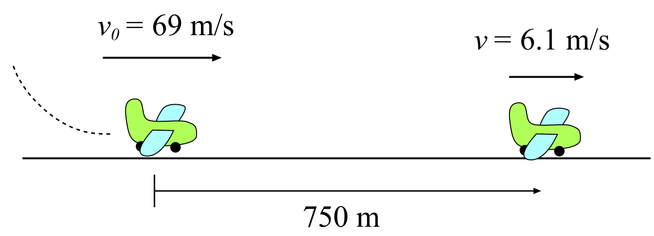

Figure 2.1: Jet landing and decreasing its speed, in Example 1.

2.2 Worked Examples

2.2.1 Motion Where the Acceleration is Constant

---

1. A jetliner, travelling northward, is landing with a speed of $69\,\frac{\mathrm{m}}{\mathrm{s}}$. Once the jet touches down, it has $750\,\mathrm{m}$ of runway in which to reduce its speed to $6.1\,\frac{\mathrm{m}}{\mathrm{s}}$. Compute the average acceleration (magnitude and direction) of the plane during landing. [CJ6 2-25]

We organize ourselves by drawing a picture of the landing plane, as shown in Fig. 2.1. The plane touches down at $x = 0$; that's where the motion begins, as far as we're concerned. The initial velocity is $v_0 = 69\,\frac{\mathrm{m}}{\mathrm{s}}$. In the final position (after it has travelled the full extent of the runway), $x = 750\,\mathrm{m}$ and $v = 6.1\,\frac{\mathrm{m}}{\mathrm{s}}$. But we are not given the time t for this motion to take placed and we don't know the (constant) acceleration a.

If we want to get a we can use Eq. 2.8, because it doesn't contain the time t. Plugging in the numbers, we get:

$$(6.1\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = (69\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + 2a(750\,\mathrm{m})$$

Do some algebra and solve for a:

$$a = \frac{(6.1\,\frac{\mathrm{m}}{\mathrm{s}})^2 - (69\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2(750\,\mathrm{m})} = -3.15\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

We get a negative answer, and we *expect* that; the plane's velocity (in the direction of motion, North) is *decreasing*. The acceleration has a *magnitude* of $3.15\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and its direction is opposite the direction of motion, i.e. South.

---

2. A drag racer, starting from rest, speeds up for $402\,\mathrm{m}$ with an acceleration of $+17.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. A parachute then opens, slowing the car down with an acceleration of $-6.10\,\frac{\mathrm{m}}{\mathrm{s}^2}$. How fast is the racer moving $3.50 \times 10^2\,\mathrm{m}$ after the parachute opens? [CJ6 2-28]

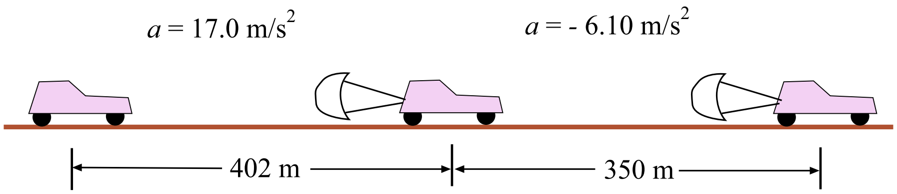

Figure 2.2: Motion of the drag racer in Example 2.

A diagram of the motion will help! This is shown in Fig. 2.2. First, let's find the velocity of the racer at the time the chute opened. We can use Eq. 2.8; with $v_0 = 0$ (the racer starts from rest), $a = +17.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and $x = 402\,\mathrm{m}$, solve for v:

$$v^2 \;=\; v_0^2 + 2ax \;=\; 2(402\,\mathrm{m})(17\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 6.83\times 10^3\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

So then

$$v = 82.7\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Now consider the part of the motion after the chute opens; we must consider it separately since the acceleration here is different from the first part of the motion. For this part of the motion the initial velocity is the value we found for the *final* velocity of the earlier motion:

$$v_0 = 82.7\,\tfrac{\mathrm{m}}{\mathrm{s}} \qquad \text{Second part of motion}$$

We have the distance covered for this part of the motion ($x = 350\,\mathrm{m}$) and the acceleration ($a = -6.10\,\frac{\mathrm{m}}{\mathrm{s}^2}$; the racer's velocity decreases during this part) and we can again use Eq. 2.8:

$$v^2 \;=\; v_0^2 + 2ax \;=\; (82.7\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + 2(-6.10\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(350\,\mathrm{m}) = 2.56\times 10^3\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

and this gives

$$v = 50.6\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The racer has a speed of $50.6\,\frac{\mathrm{m}}{\mathrm{s}}$ when it has moved $350\,\mathrm{m}$ past the point where the chute opened.

2.2.2 Free-Fall

---

3. A penny is dropped from the top of the Sears Tower in Chicago. Considering that the height of the building is $427\,\mathrm{m}$ and ignoring air resistance, find the speed with which the penny strikes the ground. [CJ6 2-37]

A picture of the problem is given in Fig. 2.3, where we've drawn the coordinate axis. The

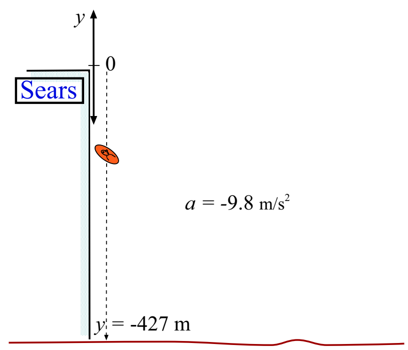

Figure 2.3: Penny dropped from top of Sears Tower in Example 3.

penny begins its motion at $y = 0$ and since it falls down, its coordinate upon striking the ground is $-427\,\mathrm{m}$. Since we *drop* the penny its initial velocity is $v_0 = 0$ and its acceleration during the fall is $a = -g = -9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

We are looking for the final velocity v but we don't have the *time* of the fall. We can use Eq. 2.8 since that equation doesn't contain t. We find:

$$\begin{aligned} v^2 &= v_0^2 + 2ax \\ &= 0^2 + 2(-9.8\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(-427\,\mathrm{m}) \\ &= 8.37 \times 10^3\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2} \end{aligned}$$

Taking the square root of this number gives $91.5\,\frac{\mathrm{m}}{\mathrm{s}}$ but there are really *two* answers for v, namely $\pm 91.5\,\frac{\mathrm{m}}{\mathrm{s}}$, and since the penny is falling *downward* when it hit the ground we want the negative one:

$$v = -91.5\,\frac{\mathrm{m}}{\mathrm{s}} \quad .$$

But the answer to the question is that the penny's *speed* (the absolute value of v) was $91.5\,\frac{\mathrm{m}}{\mathrm{s}}$ when it hit the ground.

---

4. From her bedroom window a girl drops a water–filled balloon to the ground, $6.0\,\mathrm{m}$ **below. If the balloon is released from rest, how long is it in the air?** [CJ6 2-41]

The problem is diagrammed in Fig. 2.4. The coordinate system is shown; the positive y axis points up, and (as always) we assume that the balloon starts its motion at $y = 0$. But if that is the case, then when the balloon hits the ground, its y coordinate is $-6.0\,\mathrm{m}$.

The initial velocity of the balloon is $v_0 = 0$ and its acceleration is $a = -g = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. To find how long the balloon is in the air, we ask the question: At what time is y equal to $-6.0\,\mathrm{m}$? We can then find t using Eq. 2.7. So we write:

$$-6.0\,\mathrm{m} = 0 + \tfrac{1}{2}(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2$$

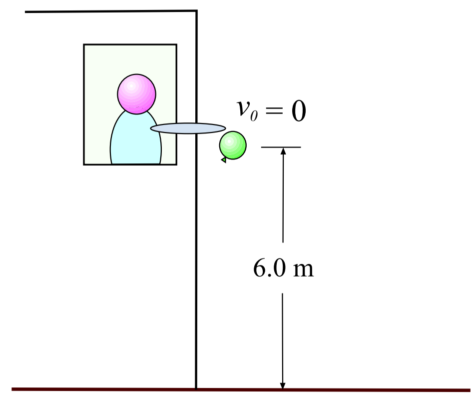

Figure 2.4: Water–balloon is dropped in Example 4.

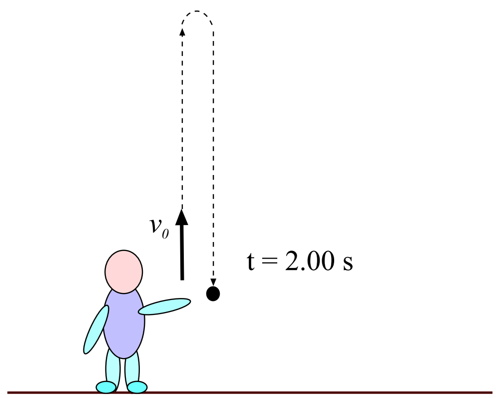

Figure 2.5: Rock is tossed up in the air in Example 5.

and solve for t. We find:

$$t^2 = \frac{2(-6.0\,\mathrm{m})}{(-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 1.22\,\mathrm{s}^2$$

and then

$$t = 1.11\,\mathrm{s}$$

The balloon hits the ground at $t = 1.11\,\mathrm{s}$, so it spends $1.11\,\mathrm{s}$ in the air.

---

5. A ball thrown vertically upward is caught by the thrower after $2.00\,\mathrm{s}$. Find (a) the initial speed of the ball and (b) the maximum height the ball reaches. [Ser7 2-48]

(a) We sketch the problem in Fig. 2.5. The ball has some initial speed v_0 (which we don't know). We know the acceleration of the ball, namely $a = -g = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. We also know that at $t = 2.00\,\mathrm{s}$ the y coordinate of the ball was zero. (As usual, we say the ball starts off at $y = 0$.) If we put that information into Eq 2.7 we get:

$$x = v_0 t + \tfrac{1}{2}at^2 \qquad \Longrightarrow \qquad 0 = v_0(2.00\,\mathrm{s}) + \tfrac{1}{2}(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(2.00\,\mathrm{s})^2$$

and now we can solve this for v_0:

$$v_0(2.00\,\mathrm{s}) = \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(2.00\,\mathrm{s})^2 = 19.6\,\mathrm{m}$$

This gives:

$$v_0 = \frac{(19.6\,\mathrm{m})}{(2.00\,\mathrm{s})} = 9.80\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

(b) We know that at maximum height the velocity v is zero. We can use Eq 2.8 to get the value of y at this time:

$$v^2 = v_0^2 + 2ay \qquad \Longrightarrow \qquad 0 = (9.80\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + 2(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})y$$

Solve this for y and get:

$$y = \frac{(9.80\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 4.90\,\mathrm{m}$$

so the maximum height attained by the ball was 4.90 m.

---

6. An astronaut on a distant planet wants to determine its acceleration due to gravity. The astronaut throws a rock straight up with a velocity of $+15\,\frac{\mathrm{m}}{\mathrm{s}}$ and measures a time of $20.0\,\mathrm{s}$ before the rock returns to his hand. What is the acceleration (magnitude and direction) due to gravity on this planet? [CJ6 2-39]

A diagram of the path of the rock is shown in Fig. 2.6. The y axis is measured upward from the position of the hand.

We know the initial velocity of the rock, $v_0 = +15.0\,\frac{\mathrm{m}}{\mathrm{s}}$ but we don't know the value of the acceleration, a_y. (We *do* know that it will be a negative number, because objects fall *down* on this planet too!) We know that at $t = 0.0\,\mathrm{s}$ y is $0.0\,\mathrm{m}$ (of course) but we also know that at $t = 20.0\,\mathrm{s}$, y is equal to $0.0\,\mathrm{s}$.

If we put the second piece of information into Eq. 2.7 we get

$$0.0 = (15.0\,\tfrac{\mathrm{m}}{\mathrm{s}})(20.0\,\mathrm{s}) + \tfrac{1}{2}a(20.0\,\mathrm{s})^2$$

from which we can find a. Some algebra gives us:

$$\tfrac{1}{2}a(20.0\,\mathrm{s})^2 = -300\,\mathrm{m}$$

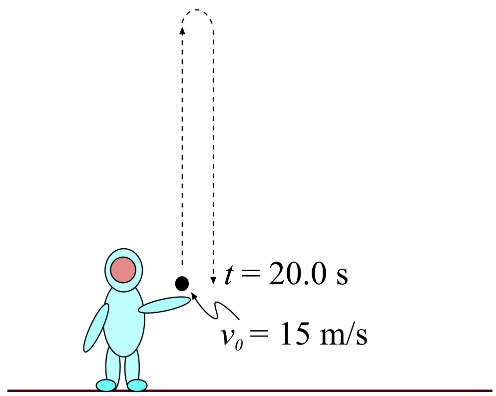

Figure 2.6: Path of tossed rock in Example 6.

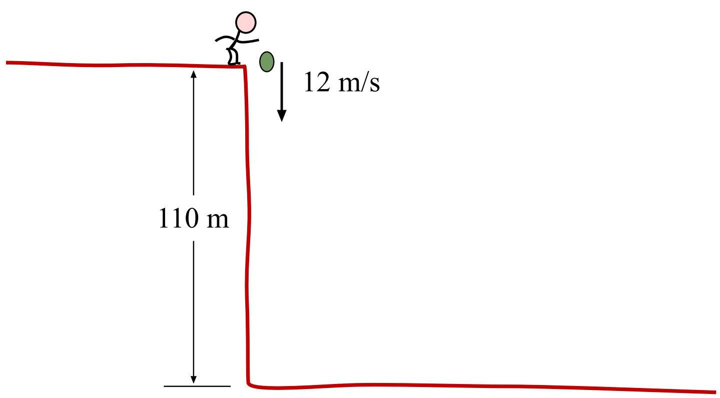

Figure 2.7: Man throws rock downward with speed $12\,\frac{\mathrm{m}}{\mathrm{s}}$, in Example 7.

Then:

$$a = -\frac{2(300\,\mathrm{m})}{(20.0\,\mathrm{s})^2} = -1.50\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The magnitude of the acceleration due to gravity on the planet is $1.50\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and from the minus sign we know that the direction of the acceleration is downward. (No surprise... things fall *down* on other planets as well!

---

7. A man stands at the edge of a cliff and throws a rock downward with a speed of $12.0\,\frac{\mathrm{m}}{\mathrm{s}}$. Sometime later it strikes the ground $110\,\mathrm{m}$ below the place where it was thrown. (a) How long does it take to reach the ground? (b) What is the speed of the rock at impact?

(a) The problem is illustrated in Fig. 2.7. Since the rock is thrown *downward*, the initial velocity of the rock is $v_0 = -12.0\,\frac{\mathrm{m}}{\mathrm{s}}$, and of course $a = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. When the rock hits the ground its y coordinate is $y = -110\,\mathrm{m}$, so in this part we are asking "At what time does $y = -110\,\mathrm{m}$?"

y is given by

$$y = v_0 t + \tfrac{1}{2}at^2 = (-12.0\,\tfrac{\mathrm{m}}{\mathrm{s}})t - \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2$$

so we just need to solve

$$-110\,\mathrm{m} = (-12.0\,\tfrac{\mathrm{m}}{\mathrm{s}})t - \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2$$

Dropping the units for simplicity, a little algebra gives

$$(4.90)t^2 + (12.0)t - 110 = 0$$

which is a quadratic equation. (Recall Eq. 1.1.) Using the quadratic formula, there are two possible answers, given by

$$t = \frac{(-12.0) \pm \sqrt{(12.0)^2 + 4(4.90)(110)}}{2(4.90)} \quad .$$

A little calculator work gives the two (?) answers:

$$t = -6.12\,\mathrm{s} \qquad \text{or} \qquad t = 3.67\,\mathrm{s}$$

So which is the answer? (There can only be one time of impact!) The answer must be the second one because a *negative* time t is meaningless; the rock was *thrown* at $t = 0$. Therefore the rock takes 3.67 s to reach the ground.

(b) We need to find the velocity of the rock at the the time found in part (a). The velocity of the rock is given by

$$v = v_0 + at = (-12\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t$$

so at $t = 3.67\,\mathrm{s}$ it is

$$v = (-12.0\,\tfrac{\mathrm{m}}{\mathrm{s}}) - (9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(3.67\,\mathrm{s}) = -18.1\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and so the *speed* of the rock at impact is $18.1\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

8. Two identical pellet guns are fired simultaneously from the edge of a cliff. These guns impart as initial speed of $30.0\,\frac{\mathrm{m}}{\mathrm{s}}$ to each pellet. Gun A is fired straight upward, with the pellet going up and falling back down, eventually hitting the ground beneath the cliff. Gun B is fired straight downward. In the absence of air resistance, how long after pellet B hits the ground does pellet A hit the ground? [CJ6 2-43]

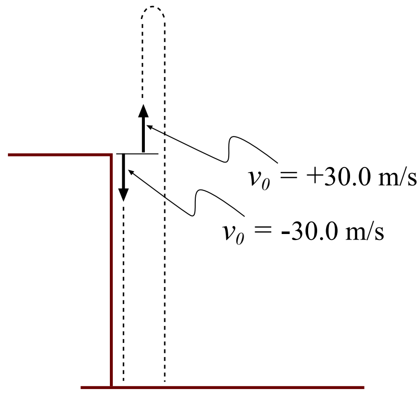

Figure 2.8: Two pellet guns shoot pellets; pellet from Gun A goes up then down. Pellet from B goes straight down.

Hoo! This one sounds complicated. And they didn't even tell us how high the cliff is! (Doesn't it matter?) We draw a picture of the problem, as in Fig. 2.8.

It turns out that if we understand something about the motion of pellet A the problem is much simpler. Let's ask: What is the velocity v of pellet A when it returns to the height at which it was thrown? Here we don't care about the time, just the distances and velocities are involved, so we want to use Eq. 2.8. When the pellet returns to the original height then $y = 0$ and so we get:

$$v^2 = v_0^2 + 0 = (+30\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 900\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

and the proper solution to this equation is

$$v = -30\,\tfrac{\mathrm{m}}{\mathrm{s}} \quad .$$

Here we choose the minus sign because the pellet is moving *downward* at that time. So when the pellet returns to the same height it has the same speed but is moving in the opposite direction.

But recall that pellet B was thrown downward with speed $30\,\frac{\mathrm{m}}{\mathrm{s}}$, that is, its initial velocity was $-30\,\frac{\mathrm{m}}{\mathrm{s}}$. So from this point on, the motion of pellet A is the *same* as that of pellet B. So from that point on it will be the same amount of time until A hits the ground. Therefore the amount of time which A spends in the air *above* that spent by B is the time it spends it takes to go up and then down to the original height. Therefore we now want to answer the question: How long does it take A to go up and back to the original height?

To answer this question we can use Eq. 2.7 with $x = 0$. We can also ask how long it take until the velocity equals $-30\,\frac{\mathrm{m}}{\mathrm{s}}$, and that will be simpler. So using Eq. 2.6 with $a = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ we solve for t:

$$-30\,\tfrac{\mathrm{m}}{\mathrm{s}} = +30\,\tfrac{\mathrm{m}}{\mathrm{s}} + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t$$

We get:

$$t = \frac{(-60\,\frac{\mathrm{m}}{\mathrm{s}})}{(-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} \;=\; 6.1\,\mathrm{s}$$

Summing up, it takes 6.1 s for pellet A to go up and back down to the original height; this is the amount of time it spends in the air *longer* than the time B is in the air. So pellet A hits the ground 6.1 s after B hits the ground.

Chapter 3

Motion in Two Dimensions

3.1 The Important Stuff

3.1.1 Motion in Two Dimensions, Coordinates and Displacement

We will now deal with more general motion, motion which does *not* take place only along a straight line.

An example of this is shown in Fig. 3.1, where a ball is thrown not straight up, but at some angle θ from the horizontal. The motion of the ball takes place in a plane and its trajectory (path) through the air happens to have the shape of a parabola.

To describe the position of the ball, we now need *two* coordinates, namely x and y, defined as shown in the figure. Here we have chosen to put the origin of the coordinate system at the place where the ball begins its motion with the positive y axis pointing "up", as in the last chapter. We will usually make this choice, although we are free to make other choices for the placement of the axes... as long as we stick with our choices!

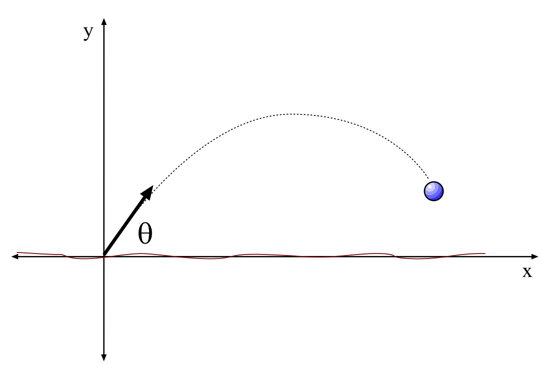

Figure 3.1: Tossed ball and coordinate system to describe its motion.

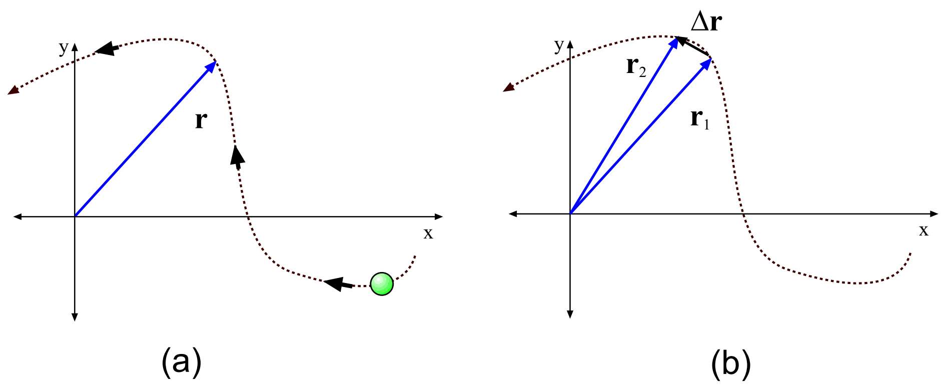

Figure 3.2: (a) Object's position is given by the displacement vector $\mathbf{r}$. (b) Change in the displacement vector as the object moves. $\Delta\mathbf{r}$ has components Δx and Δy.

The coordinates of the ball, (x and y) are the two components of the **displacement vector**, which we will write as $\mathbf{r}$. As the ball moves, the displacement vector changes. In Fig. 3.2(b) we show a change in location for an object. The displacement vector changes from $\mathbf{r}_1$ to $\mathbf{r}_2$, resulting in the change $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The components of $\Delta\mathbf{r}$ are Δx and Δy.

3.1.2 Velocity and Acceleration

Whereas in the previous chapter we only had *one* coordinate changing with time, now we have *two*: x and y. In a time interval Δt both coordinates will change.

We can now study the ratio of Δx to Δt and the ratio of Δy to Δt. These ratios are the **average** x **and** y **velocities** for the interval Δt:

$$\overline{v}_x = \frac{\Delta x}{\Delta t} \qquad \overline{v}_y = \frac{\Delta y}{\Delta t} \tag{3.1}$$

As before, the *really* interesting quantity, as far as physics is concerned, is (are) the *instantaneous* x and y velocities. These are the velocities we compute when the time interval is extremely small... as small as we can imagine:

$$\overline{v}_x = \frac{\Delta x}{\Delta t} \text{ for small } \Delta t \qquad \overline{v}_y = \frac{\Delta y}{\Delta t} \text{ for small } \Delta t \tag{3.2}$$

These equations define the x and y velocities v_x and v_y *at a particular point in time*. These velocities can change with time, and the rate of change of these velocities are the accelerations: The x and y accelerations, respectively.

v_x and v_y are the $x-$ and $y-$ components of the **velocity vector**. The magnitude of the velocity vector,

$$v = \sqrt{v_x^2 + v_y^2} \tag{3.3}$$

is called the (instantaneous) *speed* of the particle. Speed is always a positive number and like velocity it has units of $\frac{\mathrm{m}}{\mathrm{s}}$.

The instantaneous x and y accelerations are defined by:

$$a_x = \frac{\Delta v_x}{\Delta t} \text{ for small } \Delta t \qquad a_y = \frac{\Delta v_y}{\Delta t} \text{ for small } \Delta t \tag{3.4}$$

and a_x and a_y are the $x-$ and $y-$ components of the **acceleration vector**.

Basically the equations given above don't involve any new it ideas from those given in the last chapter. What *is* new is the fact that we are finding these quantities (velocity and acceleration) for the x and y coordinates *separately*, and in our problem solving we will have to think about both coordinates at once, so the problems will generally be more challenging.

3.1.3 Motion When the Acceleration Is Constant

Though one can study all kinds of two–dimensional motion at this point, we will have enough trouble on our hands if we just settle for the simple case when *both* of the acceleration components are constant. In that case, both components of the velocity will change uniformly with time. Suppose at time $t = 0$ the velocity components v_x and v_y have the values v_{0x} and v_{0y}. (These are the **initial values** of the velocity components.) Then the values of v_x and v_y later on will be given by

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t \tag{3.5}$$

These equations have the same *form* but they are really *different* equations because in general a_x and a_y will have different values in a physics problem; v_{0x} and v_{0y} will also be different.

If we want to find the value of the coordinates x and y at time t (assuming the particle starts from the origin, $x = 0$ and $y = 0$ at time $t = 0$) then we can use:

$$x = v_{0x}t + \tfrac{1}{2}a_x t^2 \qquad y = v_{0y}t + \tfrac{1}{2}a_y t^2 \tag{3.6}$$

Again, these equations look alike but they pertain to the two parts of a particle's motion: The horizontal (x) part and the vertical (y) part.

Just as in the one-dimensional case we have an equation relating v, a and x but not containing the time t:

$$v_x^2 = v_{0x}^2 + 2a_x x \qquad v_y^2 = v_{0y}^2 + 2a_y y \tag{3.7}$$

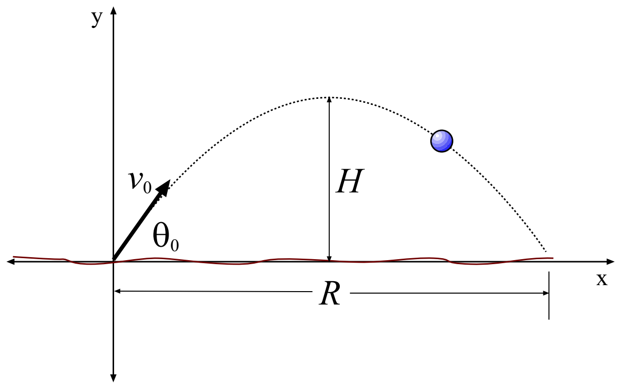

Figure 3.3: A special projectile problem; projectile is fired at angle θ_0 and initial speed v_0.

3.1.4 Free Fall; Projectile Problems

When an object is moving freely (e.g. it has been thrown or is dropped) near the surface of the earth, it undergoes a downward acceleration of magnitude $g = 9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. This means that if our y axis points *upward* then

$$a_x = 0 \qquad \text{and} \qquad a_y = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2} = -g$$

The horizontal acceleration here is *zero*... things don't fall sideways! But the vertical acceleration is $-g$... things *do* fall down!

Again, the symbol g in these notes stands for $+9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

Since the horizontal acceleration is zero, the x component of the velocity stays the same all through the motion, i.e. $v_x = v_{0x}$ during the flight of the projectile.

3.1.5 Ground–To–Ground Projectile: A Long Example

In this section we solve a special case for a projectile, the case where the projectile begins and ends its motion at the *same height*. We will get some results which are interesting and can be used in some problems, but one must keep in mind that if we have a projectile whose initial and final heights are not the same, these results are not relevant!

The derivation given here involves more math than usual for these notes, but again, the result is interesting enough that it is worth it.

We consider the motion of a projectile fired from ground level, as shown in Fig. 3.3, at angle θ_0 upward from the horizontal and a speed v_0. The projectile goes up and comes back down, striking the ground at the *same level* at which it was fired. We are interested in finding how long the projectile was in flight, the horizontal distance it travels (called the **range**, R) and its maximum height H. We are treating v_0 and θ_0 as if that are already known so that R and H will be expressed in terms of these values.

From the magnitude and direction of the initial velocity vector $\mathbf{v_0}$ we get the components of the initial velocity:

$$v_{0x} = v_0 \cos\theta_0 \qquad v_{0y} = v_0 \sin\theta_0$$

We will first answer the question: How long is the projectile in flight? That is the same as asking: "At what time does y equal zero"? Since $a_y = -g$, the y part of Eq. 3.6 gives

$$y \;=\; 0 \;=\; (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2$$

for which we can factor the right hand side to get

$$0 = t\left(v_0 \sin\theta_0 - \frac{gt}{2}\right)$$

There are two solutions to this equation. These are:

$$t = 0 \qquad \text{or} \qquad \frac{gt}{2} = v_0 \sin\theta_0 \quad \Rightarrow \quad t = \frac{2v_0 \sin\theta_0}{g}$$

The first of these possibilities is a correct answer to the question but not the one we want! The second solution gives us the time of impact:

$$t = \frac{2v_0 \sin\theta_0}{g} \tag{3.8}$$

To find the range R we ask: "What is the value of x at the time of impact?". Use the result in Eq. 3.8 and the x part of Eq. 3.6 (remembering that $a_x = 0$ for a projectile!):

$$\begin{aligned} x &= (v_0 \cos\theta_0)t - \tfrac{1}{2}a_x t^2 && (3.9)\\ &= (v_0 \cos\theta_0)\left(\frac{2v_0 \sin\theta_0}{g}\right) - 0 && (3.10)\\ &= \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} && (3.11) \end{aligned}$$

This answer can be made a little simpler using a formula from trigonometry,

$$\sin 2\theta_0 = 2 \sin\theta_0 \cos\theta_0$$

so our result is

$$R = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} = \frac{v_0^2 \sin 2\theta_0}{g} \tag{3.12}$$

Two interesting features of this solution can be noted:

• If we have a definite speed v_0 with which to launch the projectile, to give it the greatest range we would choose a launch angle of $\theta_0 = 45°$. This is because $45°$ makes the factor $\sin\theta_0$ the greatest.

• For a given launch speed, if we launch the projectile at either one of a pair of complementary angles the range R will be the same. (For example, $\theta_0 = 30°$ and $\theta_0 = 60°$ will give the same range R.) This is because for complementary angles, $\sin 2\theta_0$ is the same.

Now we'll find the maximum height of the projectile. The projectile reaches maximum height when its $y-$ velocity is zero (it is instantaneously moving neither upward nor downward at that point) so the y part of Eq. 3.5 gives:

$$v_y = 0 = (v_0 \sin\theta_0) - gt \qquad \Rightarrow \qquad t = \frac{v_0 \sin\theta_0}{g}$$

which, you'll note, is *half* the total time spent in flight. Thus the project takes as much time to go up as it does to come down.

The maximum height is the value of y at this time. Using the y part of Eq. 3.6, with $a_y = -g$, we find:

$$\begin{aligned} y &= v_{0y}t + \tfrac{1}{2}a_y t^2 = (v_0 \sin\theta_0)\left(\frac{v_0 \sin\theta_0}{g}\right) - \tfrac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2 \\ &= \frac{v_0^2 \sin^2\theta_0}{g} - \frac{v_0^2 \sin^2\theta_0}{2g} = \frac{v_0^2 \sin^2\theta_0}{2g} \end{aligned}$$

So the maximum height attained by the projectile is

$$H = \frac{v_0^2 \sin^2\theta_0}{2g} \tag{3.13}$$

Finally we can find the shape of the ball's trajectory; we can find this by relating x and y for the motion of the ball and looking at the relation that we find. Our equations for x and y were:

$$x = (v_0 \cos\theta_0)t \qquad \text{and} \qquad y = (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2 \tag{3.14}$$

The first of these gives

$$t = \frac{x}{v_0 \cos\theta_0} \ .$$

Substitute this into the second of the equations in 3.14 and do some algebra; we get:

$$\begin{aligned} y &= (v_0 \sin\theta_0)\left(\frac{x}{v_0 \cos\theta_0}\right) - \tfrac{1}{2}g\left(\frac{x}{v_0 \cos\theta_0}\right)^2 \\ &= (\tan\theta_0)x - \left(\frac{g}{2v_0^2 \cos^2\theta_0}\right)x^2 \end{aligned} \tag{3.15}$$

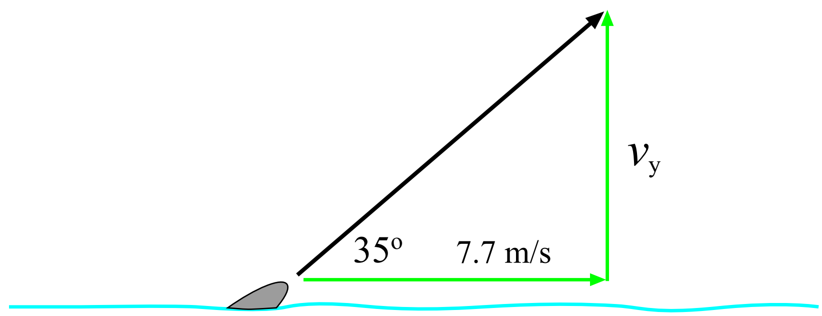

Figure 3.4: Velocity vector for jumping dolphin in Example 1.

Now while the result in Eq. 3.15 may look like a mess, the important thing to see is that there is a y on one side of the equation and an x^2 and x on the other side of the equation. From your usual college algebra class we know that this relation maps out a *parabola*. In general the trajectory of a ball tossed through the air is a parabola.

3.2 Worked Examples

3.2.1 Velocity and Acceleration

---

1. A dolphin leaps out of the water at an angle of $35°$ above the horizontal. The horizontal component of the dolphin's velocity is $7.7\,\frac{\mathrm{m}}{\mathrm{s}}$. Find the magnitude of the vertical component of the velocity. [CJ6 3-7]

The velocity vector for the dolphin is drawn in Fig. 3.4. If the vertical component of the velocity is v_y, then from trigonometry we know that:

$$\tan 35° = \frac{v_y}{v_x} = \frac{v_y}{(7.7\,\frac{\mathrm{m}}{\mathrm{s}})}$$

And then we can find v_y:

$$v_y = (7.7\,\tfrac{\mathrm{m}}{\mathrm{s}})\tan 35° = 5.4\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The vertical component of the velocity is $5.4\,\frac{\mathrm{m}}{\mathrm{s}}$. (That is also its *magnitude*.

3.2.2 Motion for Constant Acceleration

---

2. On a spacecraft, two engines are turned on for $684\,\mathrm{s}$ at a moment when the velocity of the craft has x and y components of $v_{0x} = 4370\,\frac{\mathrm{m}}{\mathrm{s}}$ and $v_{0y} = 6280\,\frac{\mathrm{m}}{\mathrm{s}}$. While the engines are firing, the craft undergoes a displacement that has components of $x = 4.11 \times 10^6\,\mathrm{m}$ and $y = 6.07 \times 10^6\,\mathrm{m}$. Find the x and y components of the craft's acceleration. [CJ7 3-12]

For the case of constant acceleration, the displacement is related to the initial velocity and acceleration by Eq. 3.6. For the x displacement we have

$$x = v_{0x}t + \tfrac{1}{2}a_x t^2$$

Plugging in the numbers from the problem, we have

$$4.11 \times 10^6\,\mathrm{m} = (4370\,\tfrac{\mathrm{m}}{\mathrm{s}})(684\,\mathrm{s}) + \tfrac{1}{2}a_x(684\,\mathrm{s})^2$$

From this we can solve for a_x. We get:

$$\tfrac{1}{2}a_x(684\,\mathrm{s})^2 = 1.12 \times 10^6\,\mathrm{m} \qquad \Longrightarrow \qquad a_x = \frac{2(1.12 \times 10^6\,\mathrm{m})}{(684\,\mathrm{s})^2} = 4.79\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

Do the same with the given values for the y displacement. Using

$$y = v_{0y}t + \tfrac{1}{2}a_y t^2$$

we have

$$6.07 \times 10^6\,\mathrm{m} = (6280\,\tfrac{\mathrm{m}}{\mathrm{s}})(684\,\mathrm{s}) + \tfrac{1}{2}a_y(684\,\mathrm{s})^2$$

Solve for a_y:

$$\tfrac{1}{2}a_y(684\,\mathrm{s})^2 = 1.77 \times 10^6\,\mathrm{m} \qquad \Longrightarrow \qquad a_y = \frac{2(1.77 \times 10^6\,\mathrm{m})}{(684\,\mathrm{s})^2} = 7.59\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration of the craft has components

$$a_x = 4.79\,\tfrac{\mathrm{m}}{\mathrm{s}^2} \qquad \text{and} \qquad a_y = 7.59\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

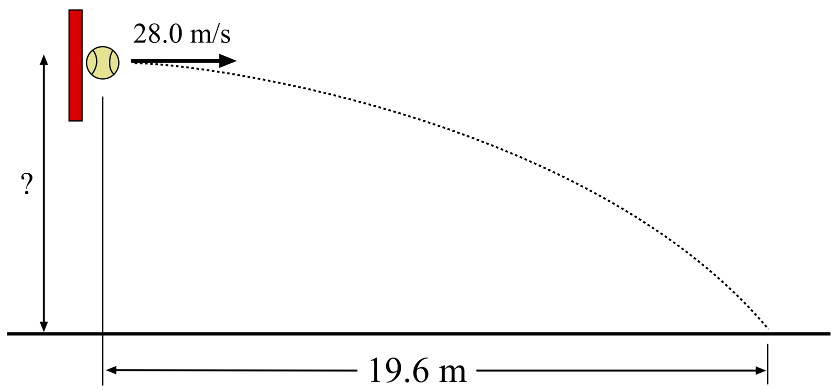

Figure 3.5: Tennis ball begins its flight horizontally in Example 3.

3.2.3 Free–Fall; Projectile Problems

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3. A tennis ball is struck such that it leaves the racket horizontally with a speed of $28.0\,\frac{\mathrm{m}}{\mathrm{s}}$. The ball hits the court at a horizontal distance of $19.6\,\mathrm{m}$ from the racket. What is the height of the tennis ball when it leaves the racket? [CJ6 3-13]

We draw picture of the ball and its motion along with the coordinates, as in Fig. 3.5.

Let's first find the time t at which the tennis ball hit the ground. We know that when it hit its x coordinate was equal to 19.6 m. Now from Eq. 3.6, the equation for the $x-$ motion is

$$x = v_{0x}t + \tfrac{1}{2}a_x t^2$$

and here the initial $x-$ velocity is $v_{0x} = 28.0\,\frac{\mathrm{m}}{\mathrm{s}}$ and $a_x = 0$ (no sideways acceleration; things fall *down*, not sideways!) The time which gives $x = 19.6\,\mathrm{m}$ is then found from:

$$19.6\,\mathrm{m} = (28.0\,\tfrac{\mathrm{m}}{\mathrm{s}})t + 0 \qquad \Rightarrow \qquad t = \frac{(19.6\,\mathrm{m})}{(28.0\,\frac{\mathrm{m}}{\mathrm{s}})} = 0.70\,\mathrm{s}$$

Now we can ask: What is the y coordinate of the ball at this time? The answer will give us the height of the ball when it was hit.

The ball's velocity at the beginning of the motion was purely horizontal, so that $v_{0y} = 0$ (no initial $y-$ velocity). The $y-$ acceleration is $a_y = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. Then the y part of Eq. 3.6 gives us:

$$\begin{aligned} y &= v_{0y}t + \tfrac{1}{2}a_y t^2 \\ &= 0 + \tfrac{1}{2}(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(0.70\,\mathrm{s})^2 = -2.4\,\mathrm{m} \end{aligned}$$

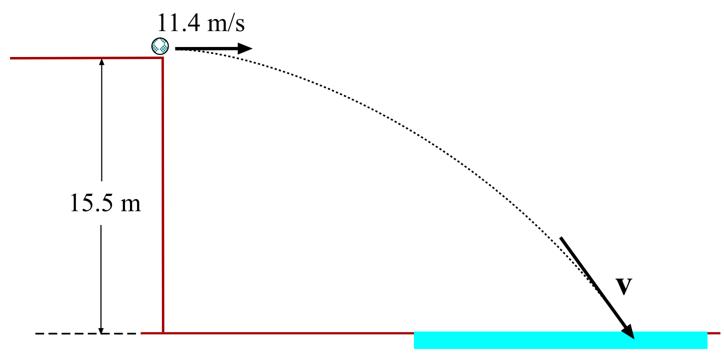

Figure 3.6: Golf ball rolls off a cliff in Example 4.

We get a negative number here because when it hits the ground, the ball has moved *downward* from its initial position of $y = 0$. But to answer the question we say that the initial height of that ball was 2.4 m.

---

4. A golf ball rolls off a horizontal cliff with an initial speed of $11.4\,\frac{\text{m}}{\text{s}}$. The ball falls a vertical distance of $15.5\,\text{m}$ into a lake below. (a) How much time does the ball spend in the air? (b) What is the speed v of the ball just before it strikes the water? [CJ6 3-15]

(a) We draw picture of the ball and its motion along with the coordinates, as in Fig. 3.6. Since the y axis goes upward, the level of the water is at $y = -15.5\,\text{m}$.

Note that the ball rolls off the cliff horizontally, so that it has an initial x velocity: $v_{0x} = 11.4\,\frac{\text{m}}{\text{s}}$, but there is no initial y velocity: $v_{0y} = 0$.

To answer (a) we think about the mathematical condition that the ball has hit the water. This is when $y = -15.5\,\text{m}$. (We don't know the x coordinate of the ball when it hits the water.) Then using the y part of Eq. 3.6 with $a_y = -g$ and v_{0y} we can solve for the time t:

$$-15.5\,\text{m} = 0 + \tfrac{1}{2}(-9.80\,\tfrac{\text{m}}{\text{s}^2})t^2$$

So

$$t^2 = \frac{2(-15.5\,\text{m})}{(-9.80\,\frac{\text{m}}{\text{s}^2})} = 3.2\,\text{s}^2$$

and then

$$t = 1.8\,\text{s}$$

(b) If we have both components of the velocity at the time the ball hits the water, we can find the speed from Eq. 3.3.

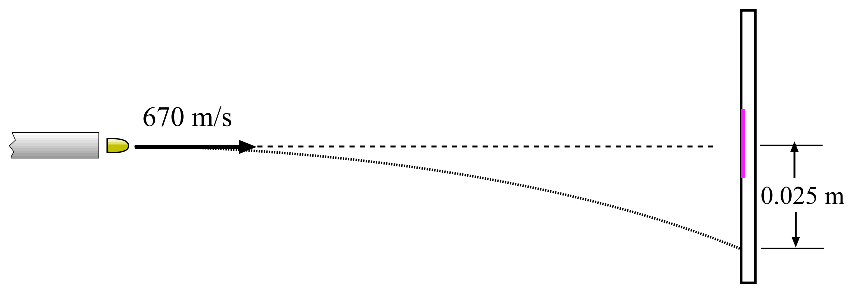

Figure 3.7: Bullet is fired horizontally at bull's–eye in Example 5.

Now since there is no $x-$ acceleration, v_x stays the same as it was at the beginning, namely $v_x = 11.4\,\frac{\mathrm{m}}{\mathrm{s}}$. Using our answer from (a) and the y part of Eq. 3.6 we find the value of v_y at impact:

$$v_y = 0 + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.8\,\mathrm{s}) = -17.4\,\tfrac{\mathrm{m}}{\mathrm{s}} \quad .$$

Then the speed of the ball at impact is

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(11.4\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-17.4\,\tfrac{\mathrm{m}}{\mathrm{s}})^2} = 20.8\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

The speed of the ball when it hits the water is $20.8\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

5. A horizontal rifle is fired at a bull's–eye. The muzzle speed of the bullet is $670\,\frac{\mathrm{m}}{\mathrm{s}}$. The barrel is pointed directly at the center of the bull's–eye, but the bullet strikes the target $0.025\,\mathrm{m}$ below the center. What is the horizontal distance between the end of the rifle and the bull's–eye? [CJ6 3-31]

As usual, begin by drawing a picture of what is happening! The problem is diagrammed in Fig. 3.7. Even though the rifle is pointed straight at the bull's–eye, the bullet *must* miss because it will take a certain amount of time to travel the horizontal distance to the target and in that time the bullet will have some downward vertical motion.

The rifle was fired horizontally, and from that we know:

$$v_{0x} = 670\,\tfrac{\mathrm{m}}{\mathrm{s}} \qquad v_{0y} = 0$$

We know that at the time the bullet struck the target its y coordinate was $y = -0.025\,\mathrm{m}$. Then using the y part of Eq. 3.6 we have:

$$-0.025\,\mathrm{m} = 0 + \tfrac{1}{2}(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2$$

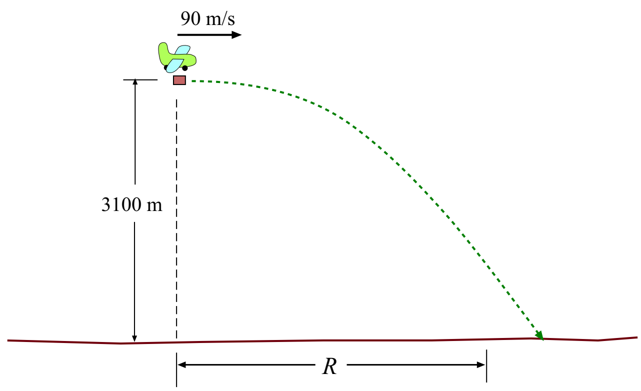

Figure 3.8: Airplane releases a package in Example 6.

which gives us

$$t^2 = \frac{2(-0.025\,\mathrm{m})}{(-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 5.1 \times 10^{-3}\,\mathrm{t}^2$$

and finally

$$t = 7.1 \times 10^{-3}\,\mathrm{s}$$

The distance to the target is the value of x at the time the bullet struck. Now that we have the time of impact we find x using the x part of Eq. 3.6:

$$x = (670\,\tfrac{\mathrm{m}}{\mathrm{s}})(7.1 \times 10^{-3}\,\mathrm{s}) = 48\,\mathrm{m}$$

---

6. An airplane is flying horizontally with a speed of $90.0\,\frac{\mathrm{m}}{\mathrm{s}}$ at an altitude of $3100\,\mathrm{m}$. The plane releases a package which falls to the level terrain below. At what distance (measured horizontally from the point of release) does the package strike the ground? Neglect air resistance!

A picture of the problem is given in Fig. 3.8. The package travels on an arcing path and eventually hits the ground. Why is this? If the package is "released", doesn't it just fall straight down? No, for reasons that can be better appreciated later on, when the package is "released" it *initially* has the velocity of its environment, namely that of the plane, and so the package has an *initial* velocity of $90.0\,\frac{\mathrm{m}}{\mathrm{s}}$. After that time though it is in free–fall and its velocity will change because of the acceleration of gravity. At impact the package has moved horizontally from its point of release by some distance R.

The package's initial velocity has only a *horizontal* component, so we have:

$$v_{0x} = 90.0\,\tfrac{\mathrm{m}}{\mathrm{s}} \qquad v_{0y} = 0$$

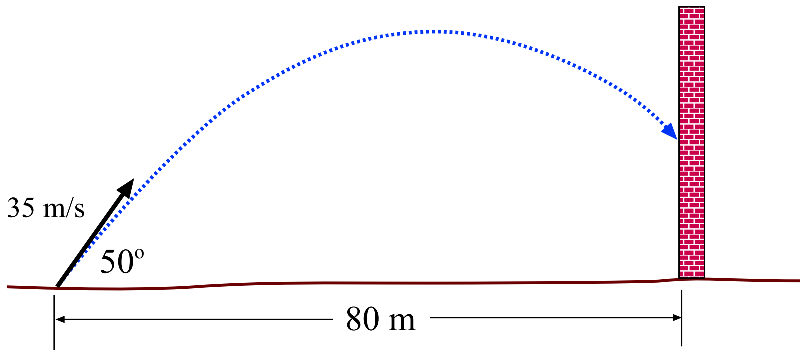

Figure 3.9: Golf ball is shot toward a tall brick wall in Example 7.

We know that y coordinate of the package when it hits the ground; that is $y = -3100\,\mathrm{m}$. (It starts at $y = 0$ and falls *downward*.) We can find the time it takes to hit the ground; find the time at which $y = -3100\,\mathrm{m}$:

$$y = v_{0y}t - \tfrac{1}{2}a_y t^2 \qquad \Longrightarrow \qquad -3100\,\mathrm{m} = 0 + \tfrac{1}{2}(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2$$

Solve for t:

$$t^2 = \frac{2(3100\,\mathrm{m})}{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 633\,\mathrm{s}^2$$
$$t = 25.2\,\mathrm{s}$$

The distance R is the value of the x coordinate at this time.

$$\begin{aligned} R &= x = v_{0x}t + \tfrac{1}{2}a_x t^2 = (90.0\,\tfrac{\mathrm{m}}{\mathrm{s}})(25.2\,\mathrm{s}) \\ &= 2.26 \times 10^3\ \mathrm{m} = 2.26\,\mathrm{km} \end{aligned}$$

At impact the package has moved a *horizontal* distance of 2.26 km from its starting point.

---

7. A golf ball is hit at a speed of $35.0\,\frac{\mathrm{m}}{\mathrm{s}}$ at 50.0° above the horizontal toward a large brick wall whose base is $80.0\,\mathrm{m}$ from the point where the ball is launched. (a) At what height does the ball strike the wall? (b) What is the speed of the ball when it hits? (c) When the ball hit the wall was it still rising or was it descending?

(a) The problem is illustrated in Fig. 3.9. (The figure shows the ball descending as it hits the wall, but that may not be the case; we need to have a *reason* for our answer to part (c).)

First find the components of the ball's initial velocity. With the initial *speed* being $v_0 = 35.0\,\frac{\mathrm{m}}{\mathrm{s}}$, we have:

$$v_{0x} = v_0 \cos\theta = (35.0\,\tfrac{\mathrm{m}}{\mathrm{s}}) \cos\,50^\circ = 22.5\,\tfrac{\mathrm{m}}{\mathrm{s}} \qquad v_{0y} = v_0 \sin\theta = (35.0\,\tfrac{\mathrm{m}}{\mathrm{s}}) \sin\,50^\circ = 26.8\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

We don't know the y coordinate for the place where the ball hits the wall, but we do know its x coordinate: It's $x = 80.0\,\mathrm{m}$. We can first find the *time* at which the ball strikes the wall by finding the time at which $x = 80.0\,\mathrm{m}$. Use the x part of Eq. 3.6 with $a_x = 0$ to get:

$$x = v_{0x}t + \tfrac{1}{2}a_x t^2 \quad \Longrightarrow \quad 80.0\,\mathrm{m} = (22.5\,\tfrac{\mathrm{m}}{\mathrm{s}})t + 0$$

Solve for t:

$$t = \frac{(80.0\,\mathrm{m})}{(22.5\,\frac{\mathrm{m}}{\mathrm{s}})} = 3.56\,\mathrm{s}$$

Now find the value of y at this time. Use the y part of Eq. 3.6 with $a_y = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and get:

$$y = v_{0y}t + \tfrac{1}{2}a_y t^2 = (26.8\,\tfrac{\mathrm{m}}{\mathrm{s}})(3.56\,\mathrm{s}) + \tfrac{1}{2}(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(3.56\,\mathrm{s})^2 = 33.3\,\mathrm{m}$$

This is the y coordinate at the time the ball hits; so the ball hits the wall at a height of 33.3 m.

(b) Use both parts of Eq. 3.5 to find the components of the velocity at the time of impact. Actually, we only need to think about the y part; since there is no x–acceleration for a projectile, v_x always keeps the same value which we found to be $22.5\,\frac{\mathrm{m}}{\mathrm{s}}$. Then:

$$v_y = v_{0y} + a_y t = 26.8\,\tfrac{\mathrm{m}}{\mathrm{s}} + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(3.56\,\mathrm{s}) = -8.1\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The speed v of the ball is the magnitude of the velocity vector, so

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(22.5\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-8.1\,\tfrac{\mathrm{m}}{\mathrm{s}})^2} = 23.9\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The ball hits the wall with a speed of $23.9\,\frac{\mathrm{m}}{\mathrm{s}}$.

(c) In part (b) we found that the y component of the velocity was *negative* at the time of impact. That tells us that the ball had *already* attained its maximum height, because maximum height is the place where $v_y = 0$. So the ball was descending at the time of impact.

---

8. The punter on a football team tries to kick a football so that it stays in the air for a long "hang time". If the ball is kicked with an initial velocity of $25.0\,\frac{\mathrm{m}}{\mathrm{s}}$ at an angle of $60°$ above the ground, what is the "hang time"? [CJ6 3-61]

Since the football begins and ends its flight at ground level, we do have the kind of projectile problem discussed in the "Ground–To–Ground" section above, and we can use the results we derived. In Eq. 3.8 we found the time in flight in terms of the launch speed and launch angle. We can use it here to get:

$$\begin{aligned} T &= \frac{2v_0 \sin\theta_0}{g} \\ &= \frac{2(25.0\,\frac{\mathrm{m}}{\mathrm{s}})}{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 5.1\,\mathrm{s} \end{aligned}$$

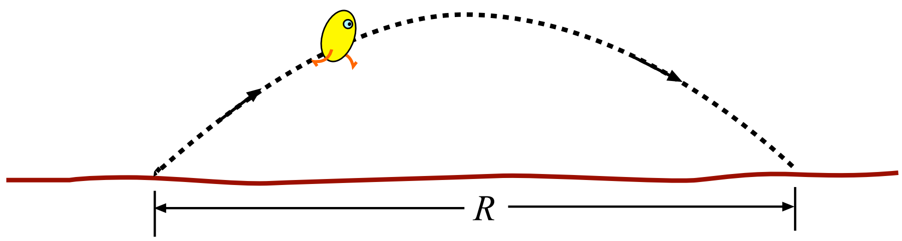

Figure 3.10: For the purpose of doing Example 9, treat the jumper as a small object.

---

9. An Olympic long jumper leaves the ground at an angle of $23°$ and travels through the air for a horizontal distance of $8.7\,\mathrm{m}$ before landing. What is the takeoff speed of the jumper? [CJ6 3-33]

Even though a jumper is not a small object, for the purpose getting an answer we will *treat* him/her as a "particle" which is launched from ground level (by its tiny legs!?!), moves through the air and then lands at ground level. See Fig. 3.10.

So this *is* a problem where the projectile (the jumper) begins and ends at the same height so we *can* use the results of the section where we got the results for the range R. We rearrange the result for R (Eq. 3.12) to solve for the initial speed v_0:

$$R = \frac{v_0^2 \sin 2\theta_0}{g} \qquad \Rightarrow \qquad v_0^2 = \frac{Rg}{\sin 2\theta_0}$$

Plug in the numbers:

$$v_0^2 = \frac{(8.7\,\mathrm{m})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{\sin(46°)} = 119\,\frac{\mathrm{m}^2}{\mathrm{s}^2}$$

So then

$$v_0 = 11\,\frac{\mathrm{m}}{\mathrm{s}}$$

Chapter 4

Forces I

4.1 The Important Stuff

4.1.1 Introduction

The preceding two chapters dealt with the *mathematics* of motion, **kinematics**. We now begin the study of the physical *reasons* for the motion of objects, i.e. the study of **dynamics**. Dynamics gives us the ability to *predict* the motion of an object in a particular physical situation.

All motion around us can be found from an application of three simple laws discovered by Isaac Newton. These laws relate the influences which govern the motion of objects (**forces**) to the accelerations of the objects and to a property of these objects, their **mass**.

The three laws are simple to state but can take years to learn how to *use*, and that is what makes physics a challenging (and interesting) subject. They were found to be adequate for describing all motion in the universe until early in the 20th century, when they were generalized by Albert Einstein to include motion at very large speeds and later by a bunch of Germans to deal with motion on the atomic scale.

Some word usage: Throughout the next few chapters we will be talking about the motion of objects whose sizes are "small" compared to the distance over which they are moving. In that case, the fact that they may be rotating will not be important and we will just discuss their overall motion. When we can treat motion in this way we will refer to the object as a **particle**.

This chapter is a long one, because in it we encounter the basis of all (classical) physical as well as examples of their usage. It is important to look at many examples of how we work with forces; only then can you get the hang of doing physics.

4.1.2 Newton's 1st Law

Newton's First Law (for our purposes) is based on earlier ideas by Galileo; it tells us what happens if we *don't* have any forces around, and states:

When there are no forces acting on an object its velocity remains the same.

This law contradicts "common sense" because we are used to all motion coming to a halt if we don't do anything to maintain it. But common experience can be deceptive, and it took the genius of Galileo and Newton to see that there are forces of friction which act on everyday objects to slow them down. Take away such influences and the motion (velocity) continues forever, unchanged.

4.1.3 Newton's 2nd Law

When there *are* influences (forces) acting on an object, then in general the velocity of the object will change, that is, there will be an acceleration. Newton's 2nd law tells us how to find that acceleration. Knowing the acceleration, we can predict the motion of the object, as covered in the preceding chapters.

When there is a stronger force acting on the object the acceleration is greater. So the acceleration is proportional to the force: $a \propto F$.

But it is also true that a given force doesn't affect all objects in the same way. If an object has more "bulk" to it, the acceleration it undergoes will be smaller; in other words a given force doesn't do as well in changing the motion of a bulky object. The proper name for property we are considering is **mass**; it is denoted by m and it is measured in kilograms. Anyway, the acceleration is *inversely* proportional to the mass of the object: $a \propto \frac{1}{m}$

Combining these two ideas, acceleration is proportional to force divided by mass: $a \propto F/m$. Multiplying both sides by m gives the simple expression of Newton's 2nd law:

$$F = ma$$

but we aren't done yet.

First, force and acceleration are both *vectors*; they have direction and magnitude. Secondly, in general there may be several forces acting on a mass m. When there are, we take the vector sum of these forces to get the total (net) force:

$$\mathbf{F}_{\rm net} = \mathbf{F}_1 + \mathbf{F}_3 + \mathbf{F}_3 + \ldots$$

and it is the net force that gives the acceleration of the object. Now we are prepared to state Newton's 2nd law:

When forces $\mathbf{F}_1$, $\mathbf{F}_2$, ... act on a mass m, the acceleration of the mass can be found from:

$$\mathbf{F}_{\text{net}} = m\mathbf{a} \quad . \tag{4.1}$$

The equation in the 2nd law is a *vector* equation so it means the same thing as when we write out the x and y components separately:

$$F_{\text{net},\,x} = ma_x \qquad \text{and} \qquad F_{\text{net},\,y} = ma_y \tag{4.2}$$

4.1.4 Units and Stuff

Mass is measured in kilograms. From the equation $\mathbf{F} = m\mathbf{a}$, we see that the units of force F must be those of mass times those of acceleration,

$$\text{Units of force} = \text{kg} \cdot \tfrac{\text{m}}{\text{s}^2} = \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$$

The combination of the basic SI units is known as the "newton", in honor of you–know–who. Thus:

$$1\,\tfrac{\text{kg}\cdot\text{m}}{\text{s}^2} = 1\,\text{newton} = 1\,\text{N}$$

The SI unit of force is the newton; sometimes we see another units of force, the **dyne**:

$$1\,\text{dyne} = 1\,\tfrac{\text{g}\cdot\text{cm}}{\text{s}^2} = 10^{-5}\,\text{N}$$

In the old "English" system of units (which we don't use in this book) the units of force is the **pound**, abbreviated as "lb" for reasons I'll never understand. The relation with the newton is:

$$1\,\text{lb} = 4.448\,\text{N}$$

4.1.5 Newton's 3rd Law

The third law of Newton is sometimes useful to us when we solve more complicated physics problems. It says something profound about nature. And it is often mis-stated and misunderstood.

Newton's Third Law says that when there is a force acting on a particle it must be due to the presence of some *other* particle(s). Forces must *come from other objects*. For there to be a (real) force acting on an object you have to be able to say what kind of object it might come from.

Secondly, if there are two objects A and B exerting forces on one another then those forces are equal in magnitude and opposite in direction. Some examples are shown in Fig. 4.1.

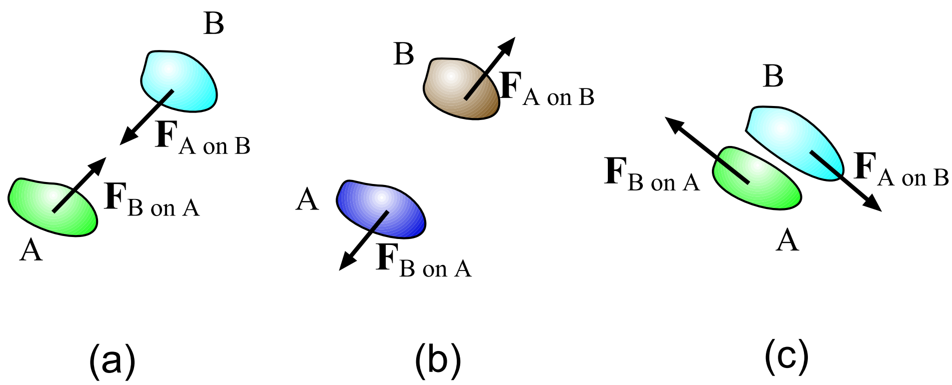

Figure 4.1: Illustrations of Newton's 3rd law. Force of A on B is "equal and opposite" to the force of B on A.

So Newton's 3rd Law is:

The force which object A exerts on object B is equal in magnitude and opposite in direction to the force which B exerts on A.

Note that this law has to do with forces; the objects may be in motion or they may not be, but the law only tells us about forces.

Unfortunately many people who think they understand physics express the law something like:

For every action there is an equal and opposite reaction

This is a poor expression of the law because we never deal with anything properly called "action" in this course, which (to my mind anyway) would seem to involve motion. It is important to understand that the third law is about *forces*, and these ("equal and opposite") forces are exerted on *two different objects*.

4.1.6 The Force of Gravity

We discussed a very important example of acceleration in the last chapter – the acceleration of a projectile – and we can now relate that to a force.

The acceleration of a projectile of mass m near the surface of the earth has magnitude g and is directed downward. Then from $\mathbf{F} = m\mathbf{a}$, there is a force on the projectile with magnitude mg and is directed downward. This is the force gravity; but where does it come from? (By Newton's 3rd law, it *must* come from another object.)

That other object is the *entire earth*. The way this comes about was also discovered by Newton. He found that all masses in the universe attract one another with a force that

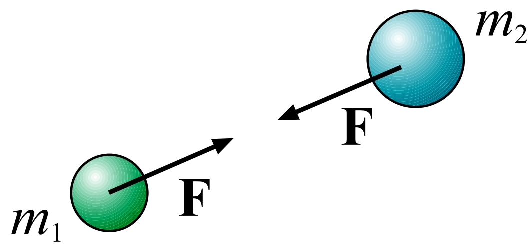

Figure 4.2: Masses m_1 and m_2 exert an attractive gravitational force on one another.

depends on the values of the masses and the distance between them; the mathematical expression is his **Law of Gravitation**.

The basic law of gravitation applies to two masses (m_1 and m_2) whose sizes are small compared to their separation (i.e. "point" masses). They are separated by a distance r. As shown in Fig. 4.2, each experiences an attractive force of magnitude F whose direction is toward the position of the other mass and whose magnitude is

$$F = G\frac{m_1 m_2}{r^2} \qquad \text{where} \qquad G = 6.67 \times 10^{-11}\,\tfrac{\text{N}\cdot\text{m}^2}{\text{kg}^2} \tag{4.3}$$

The number G is called the **gravitational constant**. Since the newton can be expressed in terms of kg, m, and s, G can also be expressed as

$$G = 6.67 \times 10^{-11}\,\tfrac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \tag{4.4}$$

There is a gravitational force of attraction between *any* two objects but for everyday objects this force is too small to be of any importance. However if one of the objects is *enormous* —like a planet— then the force is not so small. The force of attraction between a 1 kg mass and the entire earth is not small.

The problem is in how we calculate the attractive force between the earth and a small object on its surface. It is reasonable that we should use $m_1 = M_{\text{earth}}$ and $m_2 = m_{\text{object}}$ in Eq. 4.3, but what should we use for the distance of separation r?

It turns out that it is *exactly* true that since the earth is a spherically symmetric sort of thing, the proper distance to use in Eq. 4.3 is the distance between the object and the center of the earth, R_{earth}; see Fig. 4.3.

Thus the magnitude of the force of gravity on an object of mass m is

$$F_{\text{grav}} = G\frac{M_{\text{earth}} m}{R^2_{\text{earth}}} = \left(\frac{GM_{\text{earth}}}{R^2_{\text{earth}}}\right) m$$

But wait! Earlier we said that the force of gravity on an object had to be mg, or gm:

$$F_{\text{grav}} = (g)m$$

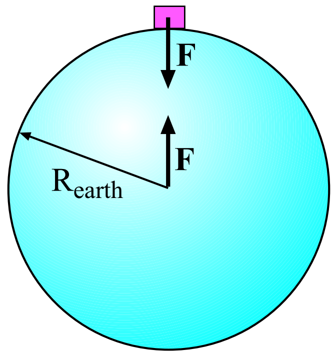

Figure 4.3: Earth exerts force F on object on its surface. For the r in Newton's law, use $R_{\rm earth}$.

Since both are true, this can only mean that

$$g = \frac{GM_{\rm earth}}{R^2_{\rm earth}} \tag{4.5}$$

Plugging in the numbers, we can see that we do get the value of g we've come to know and love:

$$g = \frac{(6.67 \times 10^{-11}\, \frac{\rm N\cdot m^2}{\rm kg^2})(5.98 \times 10^{24}\,{\rm kg})}{(6.38 \times 10^6\,{\rm m})^2} = 9.80\,\frac{\rm m}{\rm s^2}$$

and we see how the numerical value of g depends on the mass and size of the earth. We will get a different value for the acceleration of gravity on the surface of another planet, and Eq. 4.5 shows us how to calculate it.

Anway... when we consider the forces acting on *any* object on the surface of the earth, we must include the force of gravity, which is called the **weight** of the object, which, if the object has mass m, has magnitude mg and is directed downward.

4.1.7 Other Forces Which Appear In Our Problems

In order to make use of knowledge of forces, we will be solving lots and lots of problems involving simple things like blocks and strings and pulleys and inclined planes. We will be calculating the forces exerted on the masses (when we know their acceleration) or the acceleration of the masses (when we know the forces exerted on them.) Solving force problems is the real guts of a physics course!

These objects will be idealized in the sense that we will approximate their behavior with a simple rule. A real piece of string or a real pulley won't behave exactly in the way we'll use; the real behavior is more complicated and difficult.

The objects we'll see are:

- **String (cord, rope)** A string may be attached to a mass or a wall as in Fig. 4.4(a) or

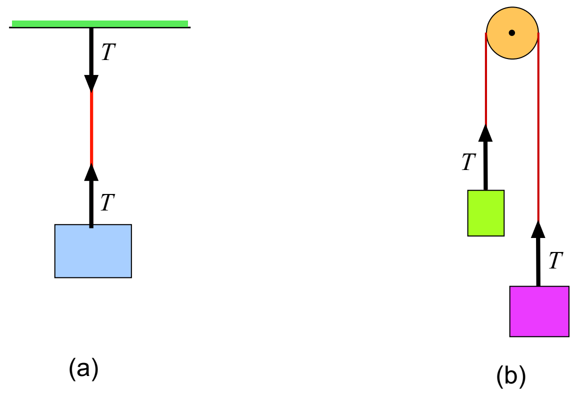

Figure 4.4: A string pulls inward along its length with a force of magnitude T.

it may be looped over a pulley, as in Fig. 4.4(b). The string will usually be *under tension*. When we are told this, it means that the string is *pulling inward at both ends* on whatever is attached with a force of magnitude T. Then T is called the **tension** in the string. (Tension is a scalar, and it has units of force, newtons.)

This is also true when the string is looped over an ideal, massless pulley, i.e. the tension is the same on both ends. Later though, we will deal with pulleys which have mass and this won't be true. For *now*, it is.

• **Smooth Surface** A mass may be in contact with a "smooth surface". We can approximate such a surface by coating it with Teflon or spraying WD–40 all over it or possibly by putting very small but ideal wheels on the block. A clever lad like you can think up something.

No real surface behaves this way because there will be a **friction force** between the surface and the mass, but we'll deal with that in the next chapter.

The force from a smooth surface is *perpendicular* to the surface. The magnitude of this force will depend on whatever is going on in the problem. This force is called the **normal force** of the surface just because in math, "normal" *means* "perpendicular".

As mentioned, in the next chapter we will deal with surfaces which are more realistic; for these there is a frictional force which points *along* the surface. But not yet; for now, the force is all normal (perpendicular).

When a mass slides on such a surface we note that its velocity and acceleration must always point along the surface, that is, there will be no component of $\mathbf{v}$ or $\mathbf{a}$ perpendicular to the surface. Since the acceleration has no component perpendicular to the surface, the component of the *net force* perpendicular to the surface must also be zero. We will use this fact in solving some problems involving hard surfaces.

• **Spring scale** Sometimes a problem will feature a **scale**. The innards of the device may not be specified but we usually mean that there's a spring of some sort inside. Two kinds of

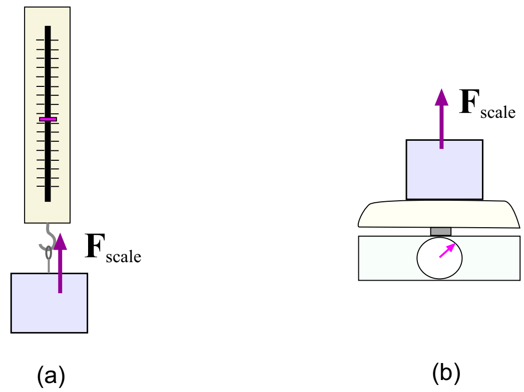

Figure 4.5: Spring scales; the spring inside the device is under some tension and pulls or pushes on the mass which is in contact with the scale. The magnitude of the force, F_{scale} is in general *not* equal to the weight of the mass!

spring scale are illustrated in Fig. 4.5. In each case the scale (or actually the spring inside of it) exerts a force on the mass with which it is in contact, and the magnitude of this force is what we read from the numerical scale on the device. In general, this value is *not* the same as the weight mg of the mass! The scale reading will depend on the details of the problem we are solving.

4.1.8 The Free–Body Diagram: Draw the Damn Picture!

In general there will several kinds of forces acting on a mass in our problems. By Newton's Second law we have to find the (vector) sum of all these force in order to get the (vector) acceleration. We will need a diagram of the directions of all the forces in order to add the vectors correctly; it's also a good to organize our thinking by noting down all the forces explicitly, since that may help us see if we've omitted any forces (or included some that shouldn't be there).

So it is always a good idea to draw a sketch showing the mass and the force vectors which act on it. Such a diagram goes by the fancy name of **free–body diagram**, but it amounts to nothing more than drawing a damn diagram of the problem (with the forces specified)!

An example is given in Fig. 4.6. In (a) we have the physical situation of the mass; in (b) we isolate the mass but show the directions of the forces acting on it.

4.1.9 Simple Example: What Does the Scale Read?

To start with a simple example, we consider a 2.0 kg mass hanging from a spring scale inside an elevator, as shown in Fig. 4.7(a). Presently, the elevator is accelerating upward at a rate of $1.80\,\frac{\text{m}}{\text{s}^2}$. What is reading on the scale?

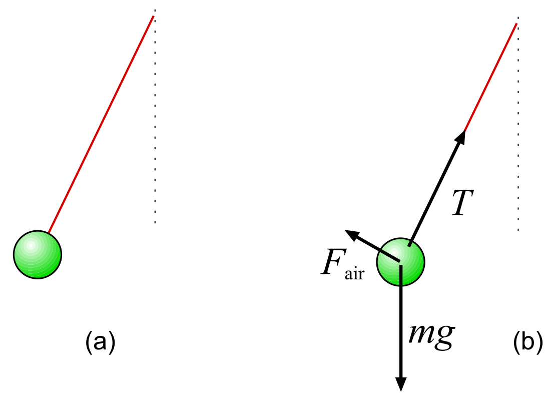

Figure 4.6: (a) A physics problem. (b) A free–body diagram for this problem.

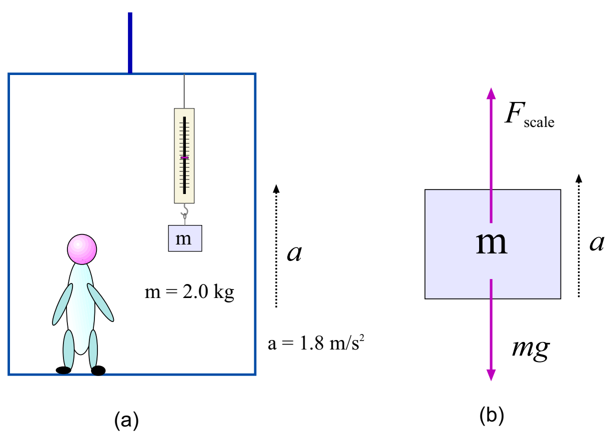

Figure 4.7: (a) Mass hangs from a scale inside an elevator which is accelerating upward. (What does the scale read?) (b) Forces acting on the mass.

It is true that if a mass m simply hangs from a scale and is *not* accelerating then the upward force of the scale $F_{\rm scale}$ will equal the downward force of gravity mg, and that is because the force vectors must add up to give zero for that case. But here the mass is accelerating along with the elevator and all its contents so there *is* a net force and the force vectors do *not* add up to zero.

In Fig. 4.7(b) we show the forces acting on the mass. Taking "up" as the $+y$ direction, the net force on the mass is

$$F_{{\rm net},\,y} = F_{\rm scale} - mg$$

and from Newton's 2nd law this is equal to ma_y, with $a_y = +1.80\,\frac{\rm m}{\rm s^2}$. Thus:

$$F_{{\rm net},\,y} = F_{\rm scale} - mg = ma_y \qquad \Longrightarrow \qquad F_{\rm scale} = mg + ma_y = m(g + a_y)$$

Now plug in the numbers and get:

$$F_{\rm scale} = (2.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2} + 1.80\,\tfrac{\rm m}{\rm s^2}) = 23.2\,{\rm N}$$

So the force of the scale (the same as the tension in its spring here) is *greater* than the value of its weight, $mg = 19.6\,{\rm N}$. The scale will *read* 23.2 N. One way to express this result is to say that the **apparent weight** of the mass is 23.2 N for the case we considered.

Now suppose the elevator car is accelerating *downward* with an acceleration of magnitude $1.80\,\frac{\rm m}{\rm s^2}$. What is the tension in the scale's spring now?

The only thing that differs from the analysis we just did is the value of a_y. Now we have $a_y = -1.80\,\frac{\rm m}{\rm s^2}$. This time we get

$$F_{\rm scale} = m(g + a_y) = (2.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2} - 1.80\,\tfrac{\rm m}{\rm s^2}) = 16.0\,{\rm N}$$

so here the force from the spring is 16.0 N so that is what the scale will read. Here the apparent weight of the mass is 16.0 N.

4.1.10 An Important Example: Mass Sliding On a Smooth Inclined Plane

An important example of the applying the principles for solving force problems is the case of a mass (or rather, a block of mass m) sliding on a frictionless inclined plane which is sloped at angle θ from the horizontal. The situation is shown in Fig.4.8. Note, we could be considering a mass which is sliding *down* the plane and moving faster and faster, or where the velocity of the mass is *up* the plane and its speed is decreasing. In either case the acceleration of the mass points *down* the slope and we would like to find its magnitude.

The first step is to identify the forces which act on the block. There are only two: One is the force of gravity, which has a magnitude mg and points straight down. The other is

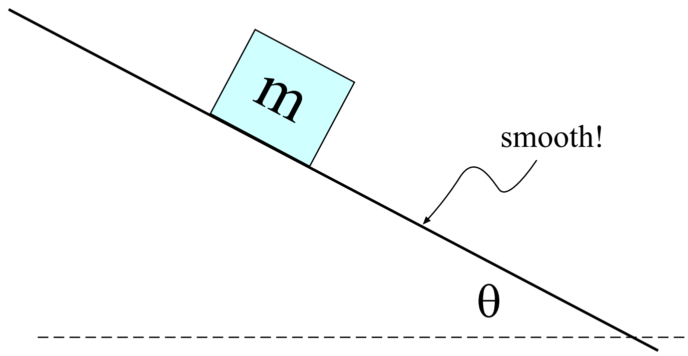

Figure 4.8: Mass m on a frictionless inclined plane. Angle of incline is θ.

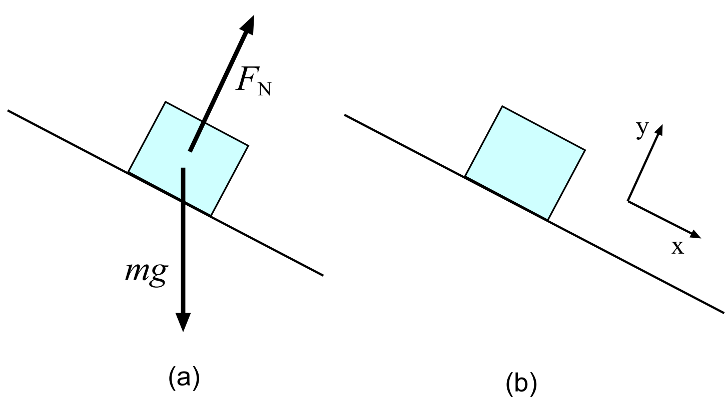

Figure 4.9: (a) Forces which act on the mass on the inclined plane. (b) We use a coordinate system with axes along the plane (x) and perpendicular to it (y).

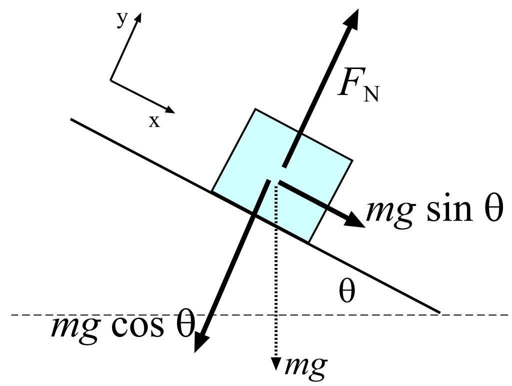

Figure 4.10: The force of gravity (downward, mg) has the indicated components in our new coordinate system.

the normal force of the surface which points perpendicular to the surface, with a magnitude we'll call F_N. These are shown in Fig. 4.9(a).

The next steps involve some reasoning and math that are a little tricky, but it important to understand them. The one thing we know about the mass is that its motion must take place along the slope of the plane. The component of its acceleration perpendicular to the plane is zero. Therefore it will be to our advantage to use a coordinate system which has axes along the plane (x, directed *down* the plane) and perpendicular to it (y). These are shown in Fig. 4.9(b). So now the normal force $\mathbf{F}_N$ points along $+y$ but the force of gravity points along neither x nor y. That's okay— we can get its x and y components in the new system and work with those instead of the original vector. One can show with some geometry that the component along the slope has magnitude $mg \sin\theta$ and the one along y (going *into* the plane) has magnitude $mg\cos\theta$. These components are illustrated in Fig. 4.10.

As we said, the y component of the total force must be zero. This gives us:

$$F_N - mg\cos\theta = 0 \qquad \text{so} \qquad F_N = mg\cos\theta$$

so (for what it's worth) we know the magnitude of the normal force from the surface. Later on, we will need this result.

The total force in the x direction is *not* zero, and Newton's 2nd Law gives us:

$$mg\sin\theta = ma_x$$

but the mass m cancels on both sides, giving us

$$a_x = g\sin\theta$$

The acceleration of the mass is directed *down* the slope (as we would expect) and it has magnitude $mg\sin\theta$. (It is a positive number because we had our x axis point *down* the slope.

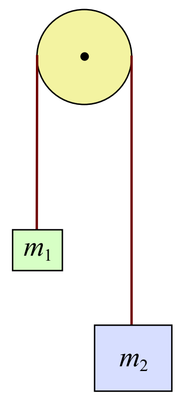

Figure 4.11: Attwood machine; if released from rest, bigger mass goes down and the smaller mass goes up!

Note that it gives the right values at $\theta = 0°$ (namely, $a_x = 0$) and at $\theta = 90°$ (namely $a_x = g$).

4.1.11 Another Important Example: The Attwood Machine

The so–called Attwood Machine consists of two masses m_1 and m_2 joined by a string which passes over a pulley. As we will treat it in this example the pulley and string are massless and ideal. Such a device is shown in Fig.4.11. If we suppose that m_2 is greater than m_1 (we don't lose any "generality" if we do that; we're always free to call the larger mass "m_2") then if the masses are released, m_2 will accelerate downward while m_1 will accelerate upward. We should be very surprised if m_1 went downward!

We would like to find the value of the accelerations of the masses and also the tension in the string.

In this problem there are two masses so we must analyze the forces (with diagrams) on both of them individually. We begin with m_1. The forces acting on m_1 are shown in Fig.4.12(a). The string tension T goes upward; the force of gravity m_1 goes downward. Now here we expect m_1 to move upward so we'll let the coordinate axis y go upward as usual and let a be the component of its upward acceleration. Then applying Newton's 2nd law in the y direction gives

$$T - m_1 g = m_1 a \qquad (4.6)$$

Now we move on to m_2. But before thinking about the forces on m_2 we can think about its motion. We expect the acceleration to be *downward* but we note that since m_1 and m_2 are connected by a taut string their motions are related. In particular at any given time their *speeds* are equal and the rates of changes of these speeds are the same. That implies that the magnitudes of their accelerations are equal. If m_1 has an acceleration a *upward*, m_2

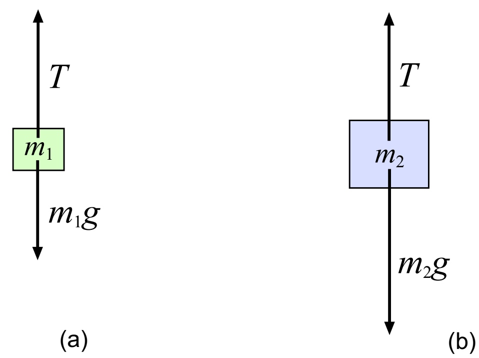

Figure 4.12: (a) Forces on m_1 in the Attwood machine. (b) Forces on m_2 in the Attwood machine.

will have an acceleration a *downward*. It will be easier for us (I think) to use a downward coordinate for m_2; then the sum of the *downward* forces will give m_2a, by Newton's 2nd law.

So now we look at the forces on m_2, shown in Fig. 4.12(b). Gravity m_2 goes downward and the string tension T (same tension on both ends) goes upward. Newton's 2nd law gives:

$$m_2g - T = m_2a \tag{4.7}$$

Now, taking m_1 and m_2 as "known" values, the two things we don't know are a and T, and we want to find these. Eqs. 4.6 and 4.7 are two equations for these two unknowns so we can use algebra to solve for them. If write them together:

$$\begin{aligned} T - m_1g &= m_1a \\ m_2g - T &= m_2a \end{aligned}$$

then add the corresponding left and right sides we cancel the tension T to get:

$$m_2g - m_1g = m_1a + m_2a$$

factoring both sides gives

$$(m_2 - m_1)g = (m_1 + m_2)a$$

and finally, dividing both sides by $(m_1 + m_2$ to isolate a gives

$$a = \frac{(m_2 - m_1)}{(m_1 + m_2)}g \tag{4.8}$$

We can *check* this result by considering two simple cases:

• If the masses are equal, $m_1 = m_2$, then Eq 4.8 gives $a = 0$, as we expect; with equal masses there is no acceleration. However in that case the masses may still have a *velocity*.

• If m_1 (the smaller mass) is *zero*, then 4.8 gives

$$a = \frac{(m_2 - 0)}{(0 + m_2)} g = \frac{m_2}{m_2} g = g$$

so m_2 falls down with the same acceleration it would have if we just dropped it. But is essentially what is happening here since we are taking m_1 and the string as having no mass. So this case gives the correct answer as well.

Now we find the tension in the string. We can use Eq. 4.6, substitute the expression for a and do some algebra:

$$\begin{aligned} T &= m_1 g + m_1 a \;=\; m_1 g + \frac{m_1(m_2 - m_1)}{(m_1 + m_2)} g \\ &= \frac{m_1(m_1 + m_2)g}{(m_1 + m_2)} + \frac{m_1(m_2 - m_1)g}{(m_1 + m_2)} \\ &= \frac{(m_1^2 + m_1 m_2 + m_1 m_2 - m_1^2)g}{(m_1 + m_1)} \\ &= \frac{2 m_1 m_2 g}{(m_1 + m_2)} \end{aligned}$$

4.2 Worked Examples

4.2.1 Newton's Second Law

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1. Forces act on a 4.0 kg mass, as shown in Fig. 4.13. Find the magnitude and direction of the acceleration of the mass when forces are as shown in (a) and (b).

(a) Here there is a single force acting in the $+x$ direction, so the net force is given by $F_{\mathrm{net},\,x} = +5.0\,\mathrm{N}$. Put this into Newton's 2nd law:

$$F_{\mathrm{net},\,x} = m a_x \qquad \Longrightarrow \qquad 5.0\,\mathrm{N} = (4.0\,\mathrm{kg}) a_x$$

Solve for a_x:

$$a_x = \frac{(5.0\,\mathrm{N})}{(4.0\,\mathrm{kg})} = 1.2\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration is in the $+x$ direction and has magnitude $1.2\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

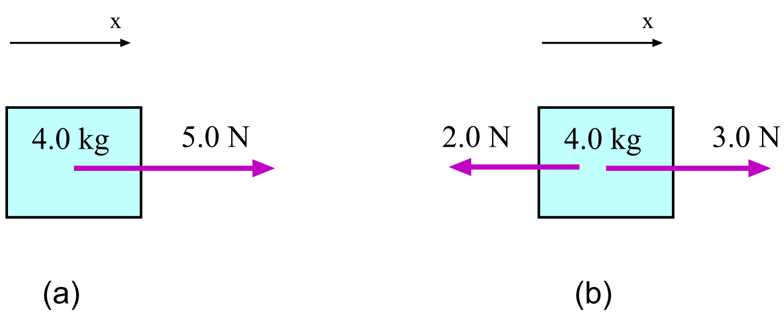

Figure 4.13: Forces act on a 4.0 kg mass in Example 1.

(b) Here we have to add two force vectors together; the forces are both directed along the x axis (one of them points in the $-x$ direction) so the total $x-$force is

$$F_{\mathrm{net},\,x} = +3.0\,\mathrm{N} - 2.0\,\mathrm{N} = +1.0\,\mathrm{N}$$

And this time Newton's 2nd law gives us:

$$a_x = \frac{F_{\mathrm{net},\,x}}{m} = \frac{(1.0\,\mathrm{N})}{(4.0\,\mathrm{kg})} = 0.25\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration of the mass is in the $+x$ direction and has magnitude $0.25\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

---

2. A car accelerates uniformly from $0\,\frac{\mathrm{mi}}{\mathrm{hr}}$ to $70\,\frac{\mathrm{mi}}{\mathrm{hr}}$ in $9.50\,\mathrm{s}$. During this time what is the force on the $80\,\mathrm{kg}$ driver?

First, find the final speed of the car in sensible units! We have:

$$70\,\tfrac{\mathrm{mi}}{\mathrm{hr}} = (70\,\tfrac{\mathrm{mi}}{\mathrm{hr}})\left(\frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right)\left(\frac{5280\,\mathrm{ft}}{1\,\mathrm{mi}}\right)\left(\frac{0.3048\,\mathrm{m}}{1\,\mathrm{ft}}\right) = 31.3\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

So the acceleration of the car is

$$a_x = \frac{v - v_0}{t} = \frac{(31.3\,\frac{\mathrm{m}}{\mathrm{s}} - 0\,\frac{\mathrm{m}}{\mathrm{s}})}{(9.5\,\mathrm{s})} = 3.29\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration of the driver is the same as that of the car! Newton's 2nd law applied to the driver gives

$$F_{\mathrm{net}\,x} = ma_x = (80.0\,\mathrm{kg})(3.29\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 264\,\mathrm{N}$$

The force on the driver (horizontal; he has no vertical acceleration) has magnitude $264\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

4.2.2 The Force of Gravity

---

3. What is the magnitude of the force of gravitational attraction between two $2.00\,\mathrm{kg}$ **masses separated by** $50.0\,\mathrm{cm}$**?**

Use Eq. 4.3 with $m_1 = m_2 = 2.0\,\mathrm{kg}$ and $r = 0.500\,\mathrm{m}$:

$$F = G\frac{m_1 m_2}{r^2} = (6.67\times10^{-11}\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}})\frac{(2.00\,\mathrm{kg})(2.00\,\mathrm{kg})}{(0.500\,\mathrm{m})^2} = 1.07\times10^{-9}\,\mathrm{N}$$

The force of gravity is $1.07\times10^{-9}\,\mathrm{N}$.

---

4. Mars has a mass of $6.46\times10^{23}\,\mathrm{kg}$ **and a radius of** $3.39\times10^{6}\,\mathrm{m}$**. (a) What is the acceleration due to gravity on Mars? (b) How much would a** $65\,\mathrm{kg}$ **person weigh on this planet?** [CJ6 4-27]

(a) Use Eq. 4.5 using the mass and radius of Mars to get g_{Mars}:

$$\begin{aligned} g_{\mathrm{Mars}} &= \frac{GM_{\mathrm{Mars}}}{R^2_{\mathrm{Mars}}} \\ &= \frac{(6.67\times10^{-11}\,\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}})(6.46\times10^{23}\,\mathrm{kg})}{(3.39\times10^{6}\,\mathrm{m})^2} \\ &= 3.75\frac{\mathrm{N}}{\mathrm{kg}} = 3.75\,\frac{\mathrm{m}}{\mathrm{s^2}} \end{aligned}$$

So we get a value which a bit more than $\frac{1}{3}$ of the value of g on the Earth.

(b) The weight of an object of mass m *on Mars* is $W_{\mathrm{Mars}} = mg_{\mathrm{Mars}}$, so

$$W_{\mathrm{Mars}} = mg_{\mathrm{Mars}} = (65\,\mathrm{kg})(3.75\,\tfrac{\mathrm{m}}{\mathrm{s^2}}) = 244\,\mathrm{N}$$

4.2.3 Applying Newton's Laws of Motion

---

5. A $3.00\,\mathrm{kg}$ **mass is pulled upward by means of an attached rope such that its acceleration is** $2.20\,\frac{\mathrm{m}}{\mathrm{s^2}}$ **upward. What is the tension in the rope?**

The basic problem is illustrated in Fig. 4.14(a). The first thing to do is to note down all the forces acting on the mass, and this done in the "free–body–diagram" in Fig. 4.14(b): The rope tension T pulls upward and the force of gravity mg is directed downward (with $m = 3.00\,\mathrm{kg}$).

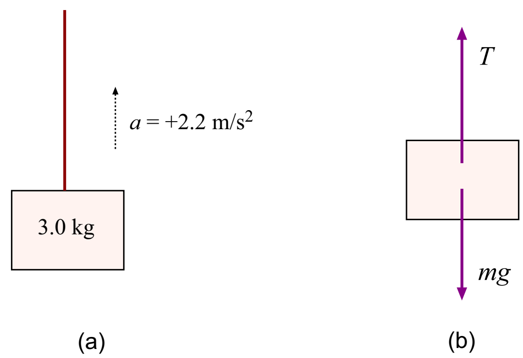

Figure 4.14: (a) Mass is pulled upward by a rope so that $a_y = +2.20\,\frac{\mathrm{m}}{\mathrm{s}^2}$. (b) Free–body–diagram for the mass.

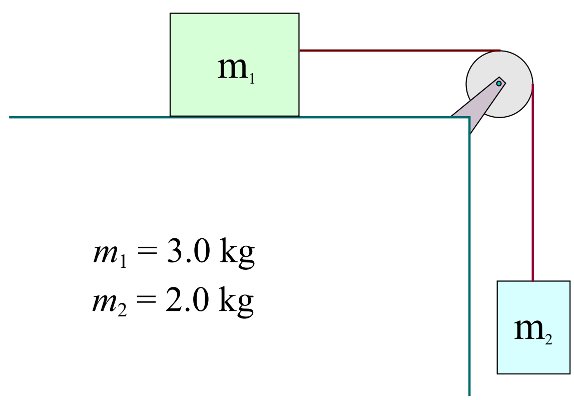

Figure 4.15: Masses joined by a string, in Example 6. m_1 (3.00 kg) slides on a smooth surface. m_2 hangs from the string and falls downward.

By Newton' 2nd law, the sum of the forces must equal $m\mathbf{a}$ and here the acceleration has magnitude $2.20\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and goes upward. So Newton's 2nd law gives:

$$T - mg = ma_y$$

Solve for T:

$$T = mg + ma_y = m(g + a_y) = (3.00\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} + 2.20\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 36.0\,\mathrm{T}$$

The tension in the rope is 36.0 N.

---

6. A 3.00 kg mass slides on a smooth horizontal surface; it is joined by a string to a hanging 2.00 kg mass. The string passes over a massless ideal pulley, as shown in Fig. 4.15. When the 2.00 kg mass is released, (a) what is the acceleration of

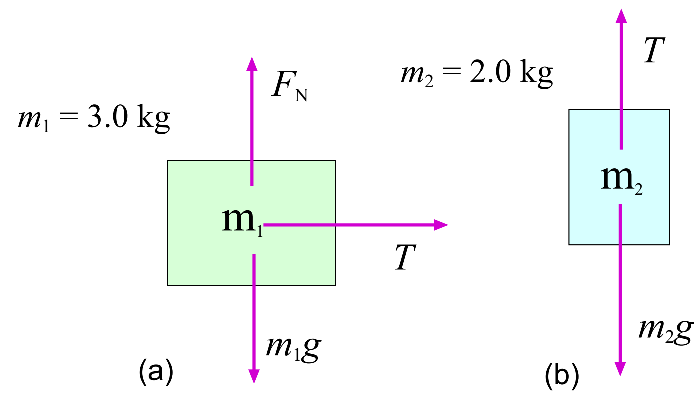

Figure 4.16: (a) The force(s) acting on m_1 in Example 6. (b) the forces acting on m_2.

the two masses? (b) What is the tension in the string?

(a) Before getting to the forces on the masses and the free–body–diagram, we ponder some features of this problem. The 3.00 kg mass (to be called m_1) will accelerate to the right and the 2.00 kg mass (m_2) will accelerate downward. These accelerations are in different directions, but since the masses are joined by a string the *distances* they travel in any amount of time are the same. So the *magnitudes* of their accelerations will be the same.

Now consider the forces acting on the masses. These are shown in Fig. 4.16. Mass m_1 experiences a force m_1g downward from gravity, a force from the table F_N (the "mormal force") upward and a force T from the string tension to the right. Now, this mass isn't moving up or down so the vertical forces have to sum to zero. This gives us

$$F_N - m_1g = 0 \qquad \text{or} \qquad F_N = m_1g$$

which is correct, but doesn't help to solve the problem.

The horizontal force on m_1 gives its acceleration (to the right) and by Newton's second law we have

$$T = m_1a \tag{4.9}$$

It is true that here $m_1 = 3.00\,\mathrm{kg}$, but it will be more useful to do some algebra *first* and then plug in the numbers at the end.

Now we look at the forces on m_2, as shown in Fig. 4.16(b). There is the string tension T pulling upward and the force of gravity m_2g pulling downward. Since we made the positive direciton of m_1's motion *to the right*, it is consistent to make the direction of positive motion for m_2 *downward*, so that is what we will do. In that case, the *downward* acceleration of m_2 is a, so adding up the *downward* forces on m_2, Newton's second law gives

$$m_2g - T = m_2a \tag{4.10}$$

Now we *know* the values of m_1 and m_2 and also the alue of g; the things we *don't* know in Eqs. 4.9 and 4.10 are T and a and of these it's really a that we want. Since we have *two* equations for *two* unknown quantities, we can do some algebra and find both of them.

If we add the respective left and right sides of the two equations we will get another (valid) equation. The reason for doing this is that the tension T will cancel on the left side. This gives:

$$T + m_2 g - T = m_1 a + m_2 a$$

A little algebra gives us

$$m_2 g = (m_1 + m_2) a$$

and then we can isolate a:

$$a = \frac{m_2 g}{(m_1 + m_2)} \ . \tag{4.11}$$

Plugging in the numbers for the particular masses, the common acceleration is

$$a = \frac{(2.00\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(3.00\,\mathrm{kg} + 2.00\,\mathrm{kg})} = 3.92\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

(b) We now have a so we can use Eq. 4.9 to get T:

$$T \;=\; m_1 a \;=\; (3.00\,\mathrm{kg})(3.92\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) \;=\; 11.8\,\mathrm{N}$$

Chapter 5

Forces II

5.1 The Important Stuff

5.1.1 Introduction

In this chapter we cover two more topics having to do with the basic application of Newton's laws.

First we discuss the force of friction, specifically as it occurs when two solid objects slide against one another. This will allow us to solve more realistic problems and give us more practice in making our force diagrams and applying Newton's 2nd law.

Then we will discuss the special (but important) case when a mass moves in a circle at constant speed. As we will see, this calls for a careful understanding of the directions of the forces and acceleration.

5.1.2 Friction Forces

Now we need to go back to the examples of the last chapter and add a little realism.

We had some examples where a mass (i.e. a block) slides along a flat surface. Here we want to take account of the fact that real surfaces are not smooth; in addition to the force perpendicular to the surface (the "normal force") there is also a force *parallel* to the surface which comes from **friction** forces.

How do we get the magnitude and direction of this friction force? Alas, it is a bit tricky but to have even the simplest discussion of friction we need to go into some details.

Sliding friction forces come in two kinds. In one kind, the block is not moving over the surface; a frictional force is *opposing some other applied forces* so that there is no net force and the block does not move. This is a force of **static friction** from the surface.

Now if we apply larger and larger forces to the block eventually it *will* move. So the static friction force can take on values up to some *maximum* value, which, in a given situation we'll call $f_{\rm s}^{\rm Max}$.

How can we find this maximal value? Empirically one finds that it depends on two things: (!) The kinds of surfaces (materials) that are rubbing together. (2) The *normal force* between the two surfaces. The maximal value is *proportional* to the normal force between the surfaces (as we might expect; push them together harder and they stick together more) so the formula for $f_{\rm s}^{\rm Max}$ turns out to be

$$f_{\rm s}^{\rm Max} = \mu_{\rm s} F_N \tag{5.1}$$

where F_N is the normal force between the surfaces and $\mu_{\rm s}$ is a number which depends on the types of surfaces which are in contact. $\mu_{\rm s}$ is called the **coefficient of static friction** and since $f_{\rm s}^{\rm Max}$ and F_N are both forces it doesn't have any units (i.e. it is pure number). It is very small when the surfaces are smooth and non–sticky.

The next kind of sliding friction happens when a block is moving over a rough surface. Now there is a force of **kinetic friction** whose direction is *opposite the direction of motion* and which has a magnitude $f_{\rm k}$. The magnitude of this friction force is given by a rule similar to the one which gave the *maximum* value of the static friction force. $f_{\rm k}$ depends on the types of materials in contact as well as the normal force between the two surfaces. Empirically, one finds:

$$f_{\rm k} = \mu_{\rm k} F_N \tag{5.2}$$

where the number $\mu_{\rm k}$ is a unitless constant called the **coefficient of kinetic friction**.

Note how Eq. 5.2 is different from Eq. 5.1. Eq. 5.2 gives the *actual value* of the force of kinetic friction whereas Eq. 5.1 just tells us how large the static friction force can possibly be; its specific value will depend on the details of the other forces acting on the mass.

5.1.3 An Important Example: Block Sliding Down Rough Inclined Plane

We go back to an "Important Example" considered in the last chapter, that of a block sliding down an inclined plane. In that chapter we assumed the plane was perfectly smooth; what happens if the plane is rough, with a coefficient of kinetic friction $\mu_{\rm k}$?

The first thing to do is to make a diagram of the forces acting on the block. In Fig. 5.1(a) we show the forces which act on the block. Gravity with magnitude mg points downward. The normal force of the surface points outward from the surface with magnitude F_N. And now there is a force of kinetic friction which is directed opposite to the motion with magnitude $f_{\rm k}$. We've said that the block will be sliding *down* the slope so the friction force points *up* the slope.

Once again, there is only motion along x so there can be no net force in the y direction. This gives us

$$\sum F_y = F_N - mg\cos\theta = 0 \qquad \Longrightarrow \qquad F_N = mg\cos\theta \tag{5.3}$$

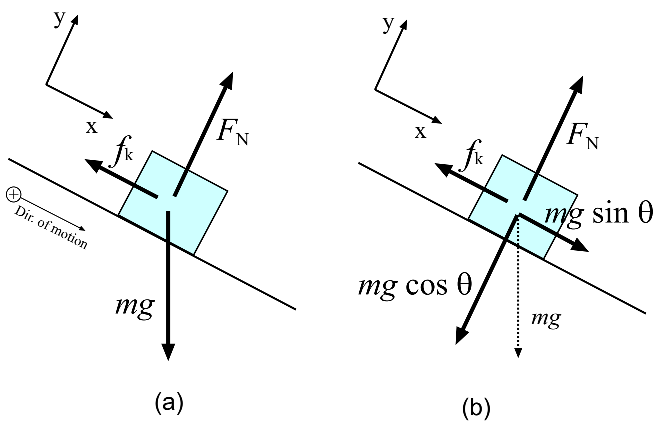

Figure 5.1: Block sliding down a rough inclined plane. (a) Basic forces acting on the block. (b) Again, it is useful to get the components of the gravity force along x and y.

and by Newton's 2nd law the sum of the x forces gives

$$\sum F_x \;=\; mg\sin\theta - f_{\rm k} \;=\; ma_x \quad . \tag{5.4}$$

From the last chapter we know that $f_{\rm k} = \mu_{\rm k} F_N$ and using Eq. 5.3 we get

$$f_{\rm k} \;=\; \mu_{\rm k} = \mu{\rm k}mg\cos\theta \tag{5.5}$$

Now using this in Eq. 5.4 we get

$$mg\sin\theta - \mu_{\rm k} mg\cos\theta = ma_x \quad .$$

We can cancel the mass m from both sides to get

$$a_x = g\sin\theta - \mu_{\rm k} g\cos\theta \tag{5.6}$$

It is interesting that once again the acceleration down the slope does not depend on the mass, but it *does* depend on the types of surfaces involved, through the coefficient of friction $\mu_{\rm k}$.

5.1.4 Uniform Circular Motion

We consider an important kind of motion not covered in Chapter 3. This is where a particle moves in a circular path (with radius r) with constant speed, v, as shown in Fig. 5.2.

Suppose it takes a time T for the particle to go around the circle. Since it then moves a distance $C = 2\pi r$ in a time T, the speed of the particle must be

$$v = \frac{2\pi r}{T} \qquad \Rightarrow \qquad T = \frac{2\pi r}{v} \tag{5.7}$$

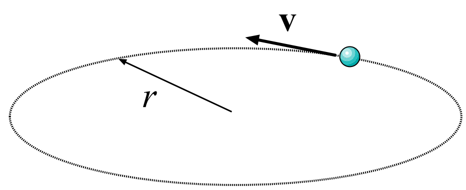

Figure 5.2: Uniform circular motion.

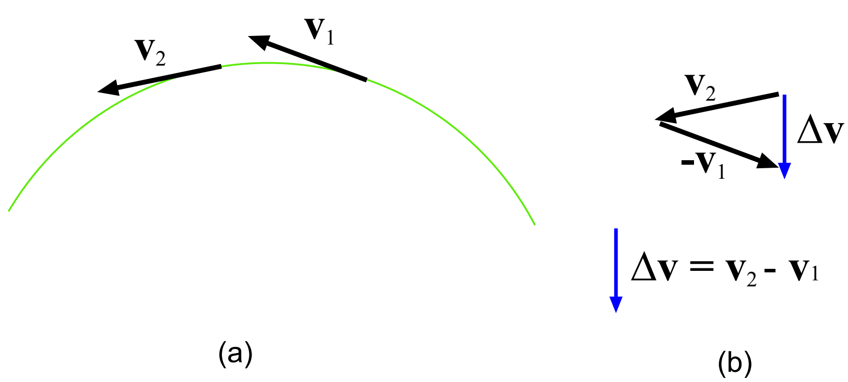

Figure 5.3: (a) Velocity vectors at two different times. (b) $\Delta\mathbf{v}$ gives the direction of the acceleration vector.

Next, some questions to test our understanding of velocity and acceleration: Is this a case of constant velocity? It is *not*. Even though the velocity of the particle always has the same *magnitude*, the *direction* of the velocity is changing, and that is important. So the particle has an acceleration and a net force which gives that acceleration.

What is the direction of this acceleration? To find the answer we have to go back to the basic definition of the acceleration *vector*; it is the rate of change in the velocity *vector*, i.e.

$$\mathbf{a} = \frac{\Delta\mathbf{v}}{\Delta t} \qquad \text{for very small } \Delta t$$

We consider the difference in velocity vectors for two points on the particle's path which are close together in time. This is shown in Fig. 5.3(a). Velocity vectors at times t_1 and t_2, $\mathbf{v}_1$ and $\mathbf{v}_2$ point very nearly in the same direction, but not quite! If we take the difference of these two vectors, as shown in Fig. 5.3(b), we find that $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ points toward the center of the circle. Since to get the acceleration we just divide $\Delta\mathbf{v}$ by the interval Δt, the acceleration must also point toward the center of the circle. Thus:

For uniform circular motion, the acceleration vector always points toward the center.

Because of this we say that the acceleration is **centripetal**. We note that since the acceleration does not have the same direction all the time (though it does have the same magnitude), the acceleration is not constant either!

What is the magnitude of the acceleration? One can show that the magnitude of the acceleration is given by:

$$a_c = \frac{v^2}{r} \tag{5.8}$$

where r is the radius of the circle and v is the (constant) speed of the particle. The subscript "c" just indicates that this is the magnitude of the *centripetal* acceleration.

5.1.5 Circular Motion and Force

Newton's 2nd law, $\mathbf{F}_{\rm net} = m\mathbf{a}$ tells us something about the forces acting on a particle which undergoes uniform circular motion. It says that the total force must also be pointing toward the center of the circle and it must have a magnitude given by

$$F_c = \frac{mv^2}{r} \tag{5.9}$$

where again the subscript c indicates that the *direction* of the force is toward the center.

5.1.6 Orbital Motion

Using the equations for uniform circular motion and Newton's law of gravity we can understand some features of the orbits of planets around the sun or of moons of the planets.

Consider a small mass m_2 in motion around a much larger mass m_1; m_1 is large enough that we will assume it is stationary. Mass m_2 experiences an attractive force toward m_1 and we will consider the case where this force provides the centripetal force for motion of m_2 around m_1 in a circular orbit of radius r. m_2 is moving with constant speed v. See Fig. 5.4.

To repeat, m_2 moves around m_1 because at all times it is being pulled *toward* m_1.

Since the distance between the two masses (their centers, actually) is r, the gravitational force on m_2 is given by

$$F_{\rm grav} = G\frac{m_1 m_2}{r^2}$$

and from the facts about its orbit, the total force on m_2 must be pointing toward the center with magnitude

$$F_c = \frac{m_2 v^2}{r}$$

But the gravity force on m_2 *is* the total force on m_2, so

$$F_{\rm grav} = F_c \qquad \Longrightarrow \qquad G\frac{m_1 m_2}{r^2} = \frac{m_2 v^2}{r}$$

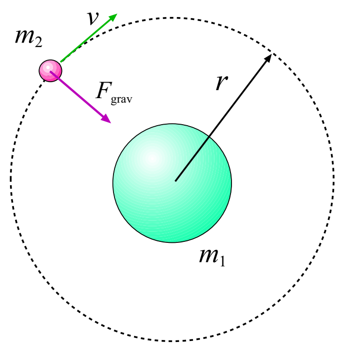

Figure 5.4: Small mass m_2 is in orbit around the much larger mass m_1; we assume that m_1 doesn't move and that the orbit is a circle centered on the center of m_1. The centripetal force on m_2 is the gravitational force $F_{\rm grav}$.

We see that m_2 cancels in the last equation as does one power of r. We can write the result as

$$Gm_1 = rv^2 \tag{5.10}$$

There is a clearer way to express Eq. 5.10 because the things we usually know about a planet or moon are the radius of the orbit r and the *period* of its motion, that is, the time it takes to make one revolution, T. Since one revolution has a path length of $2\pi r$, the speed v and T are related by

$$v = \frac{2\pi r}{T}$$

Put this into Eq. 5.10 and with a little algebra get:

$$Gm_1 = r\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 r^3}{T^2}$$

which we can write as

$$T^2 = \frac{4\pi^2 r^3}{Gm_1} \tag{5.11}$$

We note that that the mass of the small orbiting object m_2 does not appear in this equation; so it says that for *any* mass orbiting m_1 in a circular orbit, the period and orbital radius are related as given in Eq. 5.11. It says that as r gets larger so does T, but the two are not proportional; rather, the *square* of T is proportional to the *cube* of r.

Relation 5.11 is the modern version of a rule for planetary motion discovered by Kepler. Kepler considered the planets of the solar system which all orbit the same object, namely the sun. If we consider two planets with individual radii and periods r_1, T_1 and r_2, T_2, then Eq. 5.11 gives

$$\frac{T_1^2}{r_1^3} = \frac{4\pi^2}{GM} \qquad \text{and} \qquad \frac{T_2^2}{r_2^3} = \frac{4\pi^2}{GM}$$

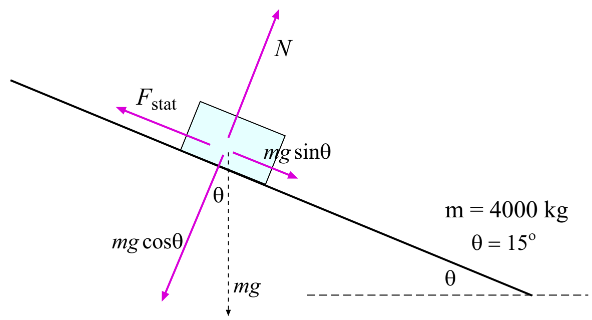

Figure 5.5: Forces acting on a 4000 kg truck is parked on a slope.

where M is the mass of the sun. This tells us:

$$\frac{T^2}{r^3} = \text{the same for all planets} \tag{5.12}$$

We can use 5.12 to get the period or distance of a planet if we know both the period and distance of *another* planet orbiting the same central object.

5.2 Worked Examples

5.2.1 Friction Forces

---

1. A 4000 kg **truck is parked on a** 15° **slope. How big is the friction force on the truck?** [KJF 5-26]

The forces which act on the truck are shown in Fig. 5.5. With $m = 4000\,\text{kg}$ and $\theta = 15^\circ$, the downward force of gravity mg has been decomposed into components along and perpendicular to the slope as we've done before. There is a normal force from the road with magnitude N. The truck has *no* acceleration so there is no net force and so there must be a force going *up* the slope, which of course is from static friction, denoted by F_{stat}.

For the forces along the slope to cancel we must have

$$F_{stat} = mg\sin\theta = (4000\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2})\sin 15^\circ = 1.01 \times 10^4\,\text{N}$$

so that is the magnitude of the friction force.

What about Eq. 5.1? That equation only gives the *maximum* value that the force of static friction can take on. No one said that this was the case in this problem. (The problem

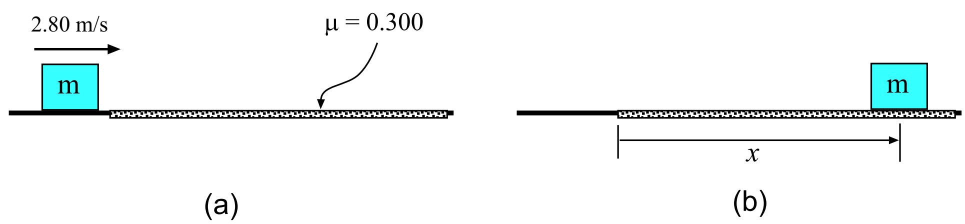

Figure 5.6: (a) Block is sliding on a smooth surface with a speed of $2.80\,\frac{\text{m}}{\text{s}}$. It encounters a surface which is rough, with $\mu_{\text{k}} = 0.300$. (b) Block has come to a halt after moving a distance x on the rough surface.

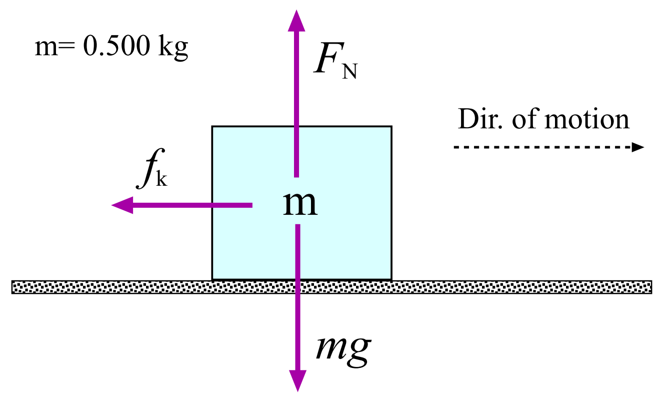

Figure 5.7: Forces acting on the mass in Example 2 while it is sliding on the rough surface.

did not say the truck is on the verge of slipping.) So while that equation is true, it's not relevant to the problem.

---

2. A block of mass $0.500\,\text{kg}$ slides on a flat smooth surface with a speed of $2.80\,\frac{\text{m}}{\text{s}}$. It then slides over a rough surface with μ_{k} and slows to a halt. While the block is slowing, (a) what is the frictional force on the block? (b) What is the magnitude of the block's acceleration? (c) How far does the block slide on the rough part before it comes to a halt?

The problem is illustrated in Fig. 5.6. As indicated in Fig. 5.6 the block slides a distance x in coming to a halt on the rough surface.

First, find the forces which act on the block... draw the damn picture, as we do in Fig. 5.7. The forces are gravity (mg, downward) the normal force from the surface (F_N, upward) and the force of kinetic friction (f_{k}, backward i.e. opposite the direction of motion).

Now, the motion only takes place along a horizontal line, so the vertical acceleration is zero. So the net vertical force on the mass is zero, giving:

$$F_N - mg = 0 \qquad \Longrightarrow \qquad F_N = mg = (0.500\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2}) = 4.90\,\text{N}$$

Now that we have the normal force of the surface, Eq. 5.2 gives us the magnitude of the (kinetic) friction force:

$$f_{\rm k} = \mu_{\rm k} F_N = (0.300)(4.90\,{\rm N}) = 1.47\,{\rm N}$$

(b) The net force on the block *is* the friction force so that the magnitude of the block's acceleration is

$$a = \frac{F_{\rm net}}{m} = \frac{(1.47\,{\rm N})}{(0.500\,{\rm kg})} = 2.94\,\tfrac{\rm m}{\rm s^2}$$

We should note that the direction of the acceleration *opposes* the diretion of motion, so if the velocity is along the $+x$ direction, the acceleration of the block is

$$a_x = -2.94\,\tfrac{\rm m}{\rm s^2}$$

Actually, by plugging the numbers into the formulae we've missed an important point. Going back to part (a), we had $F_N = mg$, so that

$$f_{\rm k} = \mu_{\rm k} F_N = \mu_{\rm k} mg$$

and the magnitude of the acceleration is

$$a = \frac{F_{\rm net}}{m} = \frac{\mu_{\rm k} mg}{m} = \mu_{\rm k} g\ ,$$

that is, the acceleration of the mass does not depend on the value of m, just on $\mu_{\rm k}$ and g.

(c) The distance travelled by the mass before it comes to a halt: We have the initial velocity v_0 of the mass, the final velocity ($v = 0$) and the acceleration. We can use Eq. 2.8 to solve for x:

$$v^2 = v_0^2 + 2ax \qquad \Longrightarrow \qquad 0^2 = (2.80\,\tfrac{\rm m}{\rm s})^2 + 2(-2.94\,\tfrac{\rm m}{\rm s^2})x$$

Solve for x:

$$x = \frac{(2.80\,\frac{\rm m}{\rm s})^2}{2(2.94\,\frac{\rm m}{\rm s^2})} = 1.33\,{\rm m}$$

---

3. A block slides down a rough incline sloped at an angle of 40.0° from the horizontal. Starting from rest, it slides a distance of $0.800\,\rm m$ down the slope in $0.600\,\rm s$. What is the coefficient of kinetic friction for the block and surface?

The problem is illustrated in Fig. 5.8. From the information given about the motion of

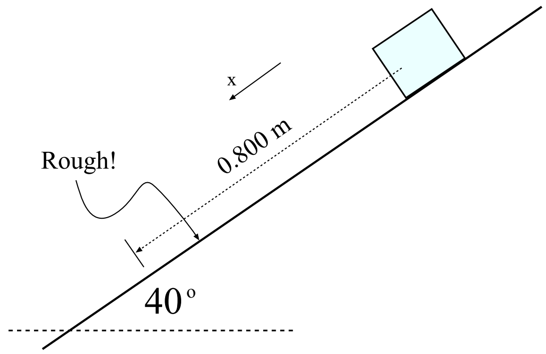

Figure 5.8: Block slides down rough inclined plane in Example 3.

the block we can find its acceleration; with the x axis pointed down the slope (as we often do in these problems), with $v_{0x} = 0$ we have:

$$x = 0 + \tfrac{1}{2}a_x t^2 \qquad \Longrightarrow \qquad a = \frac{2x}{t^2}$$

Plug in the numbers and get a_x:

$$a = \frac{2(0.800\,\mathrm{m})}{(0.600\,\mathrm{s})^2} = 4.44\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

We've solved the general problem of a block sliding down a rough inclined plane; In Eq. 5.6 we found:

$$a_x \;=\; g\sin\theta - \mu_{\mathrm{k}} g\cos\theta \;=\; g(\sin\theta - \mu_{\mathrm{k}}\cos\theta)$$

where θ is the angle of the incline. Since μ_{k} is the only thing we don't know here, we can do some algebra and solve for it:

$$\sin 40^\circ - \mu_{\mathrm{k}}\cos 40^\circ = \frac{a_x}{g} = \frac{(4.44\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 0.454$$

$$\mu_{\mathrm{k}}\cos 40^\circ = \sin 40^\circ - 0.454 = 0.189$$

$$\mu_{\mathrm{k}} = \frac{(0.189)}{\cos 40^\circ} = 0.247$$

So we get a coefficient of friction of 0.247 for the block sliding on the surface.

5.2.2 Uniform Circular Motion

4. A small mass moves on a circular path of radius $2.50\,\mathrm{m}$ with constant speed; it makes one revolution every $3.20\,\mathrm{s}$. (a) What is the speed of the mass? (b) What is the magnitude of its centripetal acceleration?

(a) Every time the mass moves around the circle, it travels a distance

$$C = 2\pi r = 2\pi(2.50\,\mathrm{m}) = 15.7\,\mathrm{m}$$

This happens in a time $T = 3.20\,\mathrm{s}$ so the speed of the mass must be

$$v = \frac{C}{T} = \frac{(15.7\,\mathrm{m})}{(3.20\,\mathrm{s})} = 4.91\,\frac{\mathrm{m}}{\mathrm{s}}$$

From Eq. 5.8 the centripetal acceleration has magnitude

$$a_c = \frac{v^2}{r} = \frac{(4.91\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(2.50\,\mathrm{m})} = 9.64\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

---

5. Find the centripetal acceleration of a point on the equator of the Earth due to the rotation of the Earth about its axis [Ser7 7-15a]

A point on the Earth's equator moves in a circle of radius

$$R_{\mathrm{earth}} = 6.38 \times 10^6\,\mathrm{m}$$

and the period of $T = 1\,\mathrm{day} = 86400\,\mathrm{s}$. So the speed of that point is

$$v = \frac{2\pi R}{T} = \frac{2\pi(6.38 \times 10^6\,\mathrm{m})}{(86400\,\mathrm{s})} = 464\,\frac{\mathrm{m}}{\mathrm{s}}$$

Then from Eq. 5.8 the centripetal acceleration of this point has magnitude

$$a_c = \frac{v^2}{R} = \frac{(464\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(6.38 \times 10^6\,\mathrm{m})} = 3.37 \times 10^{-2}\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

(The *direction* of its acceleration is toward the center of the Earth, i.e. downward.)

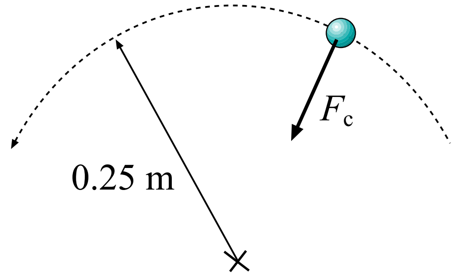

Figure 5.9: Ball moves in a circular arc.

5.2.3 Circular Motion and Force

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6. A $0.015\,\text{kg}$ **ball is shot from the plunger of a pinball machine. Because of a centripetal force of** $0.028\,\text{N}$**, the ball follows a circular arc whose radius is** $0.25\,\text{m}$**. What is the speed of the ball?** [CJ6 5-11]

A picture of the ball moving on its circular path is given in Fig. 5.9. The ball moves in a circular path because the net force on the ball *points toward the center of the circle* and has magnitude

$$F_{\text{net}} = F_{\text{c}} = \frac{mv^2}{r}$$

Since we are given F_{c}, m and r, we can solve for v:

$$v^2 = \frac{F_{\text{c}}r}{m} = \frac{(0.028\,\text{N})(0.25\,\text{m})}{(0.015\,\text{kg})} = 0.47\,\frac{\text{m}^2}{\text{s}^2}$$

So then

$$v = 0.68\,\frac{\text{m}}{\text{s}}$$

---

7. A $200\,\text{g}$ **block on a** $50\,\text{cm}$**–long string swings in a circle on a horizontal frictionless table at** $75\,\text{rpm}$**. (a) What is the speed of the block? (b) What is the tension in the string?** [KJF 6-14]

(a) The motion of the block is shown in Fig. 5.10. We are given the number of revolutions the block makes in one minute; how do we get the speed for that? We note that in each revolution the block travels a distance

$$C = 2\pi r = 2\pi(0.50\,\text{m}) = 3.14\,\text{m}$$

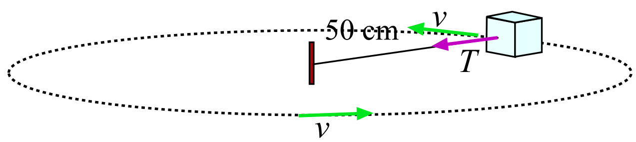

Figure 5.10: Block swings in a circular horizontal path.

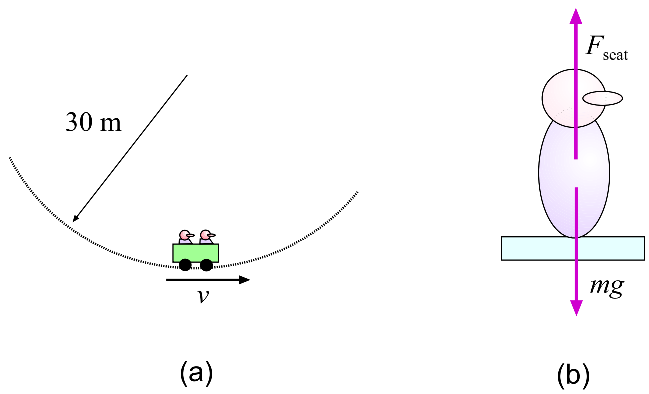

Figure 5.11: (a) Roller coaster with passengers at a dip. (b) The forces acting on a passenger at the bottom of the dip.

and if the block travels 75 times this distance in minute its speed must be

$$v = \frac{75(3.14\,\mathrm{m})}{60.0\,\mathrm{s}} = 3.92\,\frac{\mathrm{m}}{\mathrm{s}}$$

(b) There is only one force acting on the block, namely that of the string tension and it pulls *inward*. The net force on the block must be $F_c = mv^2/r$ (inward) so that gives us

$$T = \frac{mv^2}{r} = \frac{(0.200\,\mathrm{kg})(3.92\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(0.50\,\mathrm{m})} = 6.2\,\mathrm{N}$$

The tension in the string is 6.2 N.

---

8. The passengers in a roller coaster feel 50% heavier then their true weight as the car goes through a dip with a 30 m radius of curvature. What is the car's speed at the bottom of the dip? [KJF6-19]

Whoa! Passengers feel heavier? Speed at the bottom of the dip? Huh?

The motion of the car with its passengers is shown in Fig. 5.11(a). We don't know the speed v but we do know that since the car and its contents has a speed v and is in circular motion where the radius of the motion is 30 m. So the acceleration of the car is *toward* *the center of the circle*, that is to say, *upward* and has magnitude $a_c = v^2/r$. (Here we assume that the only forces on the car and passengers are vertical; there is no tangential acceleration) So the acceleration of any one of the passengers *also* has magnitude v^2/r and is directed upward.

So if the mass of a passenger is m then the net force on that passenger must have magnitude mv^2/r and be directed *upward*.

The forces acting on a passenger are shown in Fig. 5.11(b). There is the force of gravity, mg, downward and also the force of the seat, $F_{\rm seat}$, which is directed upward. These forces do *not* have equal magnitude because as we just said, the net force has magnitude $\frac{mv^2}{r}$ and is directed upward. So using our force diagram, Newton's 2nd law gives

$$F_{\rm seat} - mg = \frac{mv^2}{r}$$

But the problem tells something about $F_{\rm seat}$. Normally when we sit in chairs we are not accelerating; then the chair's force is equal to that of gravity (the weight): $F_{\rm seat} = mg$. We *feel* this force of the chair. For the passengers in the car $F_{\rm seat}$ is larger; the problem tells us that the seat's force is 50% larger than that of gravity:

$$F_{\rm seat} = (1.50)F_{\rm grav} = 1.50mg$$

and using this fact in the first equation gives

$$(1.50)mg - mg = (0.50)mg = \frac{mv^2}{r}$$

which gives (cancel the m):

$$v^2 = (0.50)gr = (0.50)(9.80\,\tfrac{\rm m}{\rm s^2})(30\,{\rm m}) = 147\,\tfrac{\rm m^2}{\rm s^2}$$

and then

$$v = 12\,\tfrac{\rm m}{\rm s}$$

---

9. In the Bohr model of the hydrogen atom, an electron (mass $m = 9.11\times10^{-31}\,\rm kg$) orbits a proton at a distance of $5.3\times10^{-11}\,\rm m$. The proton pulls on the electron with an electric force of $8.2\times10^{-8}\,\rm N$. How many revolutions per second does the electron make? [KJF 6-38]

A picture of this model for the hydrogen atom is shown in Fig. 5.12. The electron orbits in a circular orbit of radius r with speed v. The only force on the electron is the electrical force which points inward and has magnitude

$$F_c = F_{\rm elec} = \frac{mv^2}{r} = 8.2\times10^{-8}\,\rm N$$

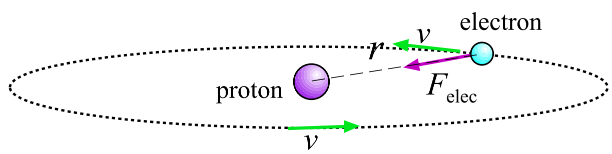

Figure 5.12: Electron orbits the proton in the Bohr model.

From this we can solve for the speed v:

$$v^2 = \frac{rF_{\rm elec}}{m} = \frac{(5.3 \times 10^{-11}\,{\rm m})(8.2 \times 10^{-8}\,{\rm N})}{(9.11 \times 10^{-31}\,{\rm kg})} = 4.8 \times 10^{12}\,\tfrac{{\rm m}^2}{{\rm s}^2}$$

so then

$$v = 2.2 \times 10^6\,\tfrac{\rm m}{\rm s}$$

This speed is about $\frac{1}{100}$ of the speed of light.

In one second the electron will travel 2.2×10^6 m, but the distance covered in one revolution is

$$C = 2\pi r = 3.3 \times 10^{-10}\,{\rm m}$$

so that the number of revolutions made in one second is

$$N = \frac{(2.2 \times 10^6\,{\rm m})}{(3.3 \times 10^{-10}\,{\rm m})} = 6.7 \times 10^{15}\,{\rm rev}$$

so the orbital rate of the electron is $6.7 \times 10^{15}\,\frac{\rm rev}{\rm s}$. This is a large number, but the situation (motion of an electron inside an atom) is rather exotic!

5.2.4 Orbital Motion

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10. One of the moons of Jupiter (Europa) orbits Jupiter in a (roughly) circular orbit with a radius of $670,900\,{\rm km}$. The period of the orbit is $3.55\,{\rm days}$. From this information, what is the mass of Jupiter?

Eventually we will use Eq. 5.11 here to get m_1 from r and T, but first we need to get things in the right units. In meters, the radius of the orbit r is

$$r = (670,900\,{\rm km})\left(\frac{10^3\,{\rm m}}{1\,{\rm km}}\right) = 6.71 \times 10^8\,{\rm m}$$

and the period is

$$T = (3.55\,{\rm day})\left(\frac{24\,{\rm hr}}{1\,{\rm day}}\right)\left(\frac{3600\,{\rm s}}{1\,{\rm hr}}\right) = 3.07 \times 10^5\,{\rm s}$$

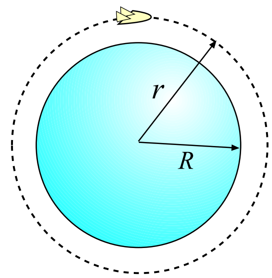

Figure 5.13: Space shuttle in orbit around the earth.

Doing some algebra on Eq. 5.11 to isolate the mass of the central body m_1 gives:

$$m_1 = \frac{4\pi^2 r^3}{GT^2}$$

and plugging in the numbers (use the form of G in Eq. 4.4) gives:

$$\begin{aligned} m_1 &= \frac{4\pi^2(6.71 \times 10^8\,\mathrm{m})^3}{\left(6.67 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg\cdot s^2}}\right)(3.07 \times 10^5\,\mathrm{s})^2} \\ &= 1.90 \times 10^{27}\,\mathrm{kg} \end{aligned}$$

We find that the mass of Jupiter is 1.90×10^{27} kg.

---

11. The space shuttle is in a 250−mile-high orbit. What are the shuttle's orbital period (in minutes) and its speed? [KJF 6-33]

We diagram the problem, as in Fig. 5.13. We need to convert 250 mi to m:

$$250\,\mathrm{mi} = (250\,\mathrm{mi})\left(\frac{10^3\,\mathrm{m}}{0.621\,\mathrm{mi}}\right) = 4.03 \times 10^5\,\mathrm{m}$$

but to find the radius of the shuttle's orbit (that is, its distance from the center of the earth) we must add this to the radius of the earth. Thus:

$$r = R + 4.03 \times 10^5\,\mathrm{m} = 6.37 \times 10^6\,\mathrm{m} + 4.03 \times 10^5\,\mathrm{m} = 6.77 \times 10^6\,\mathrm{m}$$

Having the radius of the orbit, and knowing the mass of the earth, we can use Eq. 5.11 to get the period of the orbit:

$$\begin{aligned} T^2 &= \frac{4\pi^2 r^3}{Gm_1} = \frac{4\pi^2(6.77 \times 10^6\,\mathrm{m})^3}{(6.67 \times 10^{-11}\,\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}})(5.98 \times 10^{24}\,\mathrm{kg})} \\ &= 3.17 \times 10^7\,\mathrm{s}^2 \end{aligned}$$

Then:

$$T = 5.54 \times 10^3\,\mathrm{s}\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) = 92.3\,\mathrm{min}$$

We can get the speed of the shuttle using Eq. 5.7:

$$v = \frac{2\pi r}{T} = \frac{2\pi(6.77 \times 10^6\,\mathrm{m})}{(5.54 \times 10^3\,\mathrm{s})} = 7.68 \times 10^3\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Chapter 6

Energy

6.1 The Important Stuff

6.1.1 Introduction

While Newton's laws form the basis of physics, experience has shown that we need some other concepts and laws to be able to deal with problems effectively. The present chapter and the next one deal with some new quantities and the rules which they obey. The quantities are **energy** and **momentum**.

6.1.2 Kinetic Energy

We make a definition of a quantity which we will soon find to be quite useful. For a particle of mass m moving at speed v, the **kinetic energy** of the particle is defined by:

$$\mathrm{KE} = \tfrac{1}{2}mv^2 \tag{6.1}$$

What sort of quantity is this? It is a *scalar*, that is, it is s single number which does not have any direction associated with it (unlike *velocity* which is a vector). It is always a positive number since m and v^2 are always positive.

The *units* of this quantity have to be those of mass multiplied by those of speed squared, that is,

$$\mathrm{kg}\cdot\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = \tfrac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2}$$

This is a new and important combination which is called the **joule**. Thus:

$$1\,\mathrm{joule} = 1\,\mathrm{J} = 1\,\tfrac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2} \tag{6.2}$$

Other units of energy sometimes seen are:

$$1\,\mathrm{erg} = 1\,\tfrac{\mathrm{g}\cdot\mathrm{cm}^2}{\mathrm{s}^2} = 10^{-7}\,\mathrm{J} \qquad \text{and} \qquad 1\,\mathrm{MeV} = 1.602\times10^{-19}\,\mathrm{J}$$

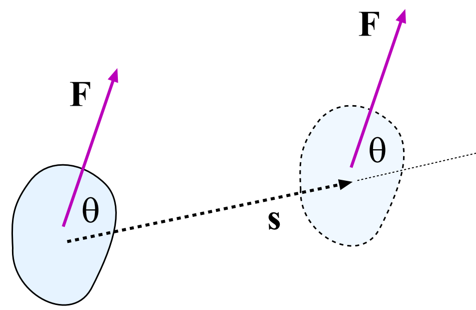

Figure 6.1: Constant force **F** acts on an object as it is displaced by **s**. Work done is $Fs\cos\theta$.

The erg is the unit of energy in the cgs system of units; the MeV is encountered when we deal with atomic processes.

6.1.3 Work

We have one more quantity to define before we say anything of any substance. The quantity is **work**. When a force acts on an object which undergoes a displacement (i.e. it moves) then work is done.

We first consider a special case for computing work. This is where the force **F** is constant (keeps the same magnitude and direction) and where the displacement of the mass is along a straight line; so the displacement (change in location **r**) is some vector **s**; see Fig.6.1

Finally, suppose the angle between the force vector **F** and the displacement **s** is θ. Then the work done by this force is

$$W = Fs\cos\theta \tag{6.3}$$

We have several things to discuss about definition 6.3 before discussing why it's important.

First off, work is a *scalar* meaning that it is a single number. The units of work are those of force times those of distance (a cosine has no units) which is the combination:

$$\mathrm{N}\cdot\mathrm{m} = \tfrac{\mathrm{kg\cdot m}}{\mathrm{s}^2}\cdot\mathrm{m} = \tfrac{\mathrm{kg\cdot m}^2}{\mathrm{s}^2}$$

which you'll notice is the very same combination that made up the units for kinetic energy, the combination that we called a “joule”. Thus, work is measured in joules.

Next we note that while F and s are both positive, the cosine of an angle can be negative (as then the angle is between 90° and 180°) so that work (W) can be negative.

Finally there are three special cases for θ that we will want to keep in mind: When the force acts in the same direction as the displacement then θ is zero, $\cos\theta = 1$ and then $W = Fs$. When the force acts in the *opposite* direction as the displacement then $\theta = 180°$, $\cos\theta = -1$ and then $W = -Fs$. When the force acts in a direction *perpendicular* to that of the displacement then $\theta = 90^\circ$, $\cos\theta = 0$ and the work done is *zero*. A force acting perpendicular to the direction of motion does *no work*.

Finally, it will be of use to find the total work done by all the forces acting on a mass. To get this, we simply add up the work done by each individual force. One can also find the net force acting on the mass and find the work done by *it*; the result is the same.

6.1.4 The Work–Energy Theorem

What are these definitions good for? One can show that the quantities kinetic energy (KE) and work (W) are related. If we consider the motion of a particle over a certain time interval we can consider the total work done on the particle, $W_{\rm tot}$. We can also consider its change in kinetic energy, which is just

$$\Delta\mathrm{KE} = \mathrm{KE_f} - \mathrm{KE_i} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}m_1^2$$

The work energy theorem tells us that they are the same:

$$W_{\rm tot} = \Delta\mathrm{KE} \tag{6.4}$$

6.1.5 Potential Energy

Some physics problems can be done using the Work–Energy Theorem but in general we need some further definitions and theorems so that it becomes really useful. In particular, calculating the work done by the various forces can be made much simpler.

One can show that when an object of mass m moves from a height y_1 to a height y_2, then work done by gravity is

$$W_{\rm grav} = -mg(y_{\rm f} - y_{\rm i}) = -mg\Delta y \ . \tag{6.5}$$

This is an interesting result because $W_{\rm grav}$ depends only on the difference in heights between the initial and final positions. The path that the particle took between the two places does not matter! The work done is just a difference in the quantity $-mgy$ evaluated between the initial and final locations.

We will see some other forces for which the work done is also expressible as a difference in some expression between the final and initial positions. Not all forces have this property but some of the more interesting ones do. Such forces are called **conservative forces** and for these forces it is easiest to find and use the expression whose difference gives W.

To be precise, it's the *negative* of the difference in some quantity which we want to equal W; the quantity is called the **potential energy**, PE. Thus, for forces which permit us to have a simple expression for the work, we define:

$$\Delta\mathrm{PE} = -W \tag{6.6}$$

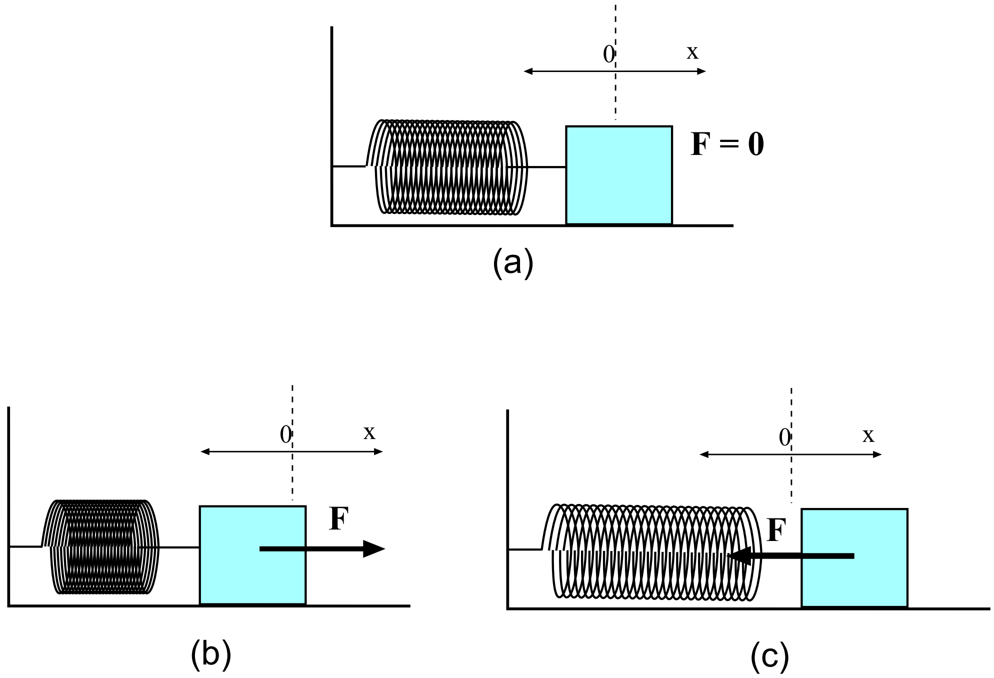

Figure 6.2: Spring exerts a force on the block which opposes the displacement of the block from the equilibrium position. (a) Spring has its natural length; no force. (b) Spring is compressed; force goes outward. (c) Spring is stretched; force goes inward.

Comparing this with Eq. 6.5, the potential energy for the gravity force (near the surface of the earth) is

$$\mathrm{PE}_{\mathrm{grav}} = mgy \tag{6.7}$$

where y is the usual vertical coordinate.

As mentioned, not all forces are conservative. The kinetic friction force does *not* have this property. As you might expect, such forces are called **nonconservative**.

6.1.6 The Spring Force

In order to have some variety at this point, I will introduce another kind of force which is important in physics. This is the force exerted by an ideal spring on something attached to its end.

In Fig. 6.2 we show the basic behavior of a spring. When it is unstretched or uncompressed it has an equilibrium length. In that case there is no force exerted on the block which is attached to one end. When it is compressed (as in (b)) or stretched (as in (c)) then a force is exerted on the block; the direction of the force is *opposite* the direction of the displacement.

Also, the spring force has a greater *magnitude* the more we compress or stretch the spring (i.e. as $|x|$ gets bigger). For the "ideal spring", the magnitude of the force is *proportional* to the amount of compression or stretch of the spring away from the equilibrium position. If we consider the motion of the mass along the x axis and if $x = 0$ corresponds to the equilibrium position then the force of the spring is given by:

$$F_x = -kx \tag{6.8}$$

where the number k is a constant and depends on the stiffness of the spring. (It is positive; the minus sign in Eq. 6.8 makes F_x opposite in sign to the displacement x.) It is called the **force constant** of the spring and since it is equal to $|F/x|$, its units are those of force divided by those of distance, or $\frac{\mathrm{N}}{\mathrm{m}}$. (This combination is also equal to $\frac{\mathrm{k}}{\mathrm{s}^2}$.)

One can show that if the mass on the end of the spring goes from x_1 to x_2, the work done by the spring force is

$$W_{\mathrm{spring}} = -\tfrac{1}{2}k(x_2^2 - x_1^2)$$

which is true regardless of how the mass gets from x_1 to x_2. Then Eq. 6.6 gives us the expression for the potential energy of the spring force:

$$\mathrm{PE}_{\mathrm{spr}} = \tfrac{1}{2}kx^2 \tag{6.9}$$

6.1.7 The Principle of Energy Conservation

Now we show why the idea of potential energy is useful.

The forces in nature are either conservative or non-conservative; when we compute the total work done by all the forces we can separate the sum into two parts, one due to the conservative forces and one due to the non-conservative forces:

$$W_{\mathrm{total}} = W_{\mathrm{cons}} + W_{\mathrm{non-cons}}$$

putting this into Eq. 6.4 we have:

$$W_{\mathrm{total}} = W_{\mathrm{cons}} + W_{\mathrm{non-cons}} = \Delta\mathrm{KE}$$

Now we note from Eq. 6.6 the work by the conservative forces is equal to the change in all the kinds of potential energy:

$$W_{\mathrm{cons}} = -\Delta\mathrm{PE}$$

Putting this into the previous equation,

$$-\Delta\mathrm{PE} + W_{\mathrm{non-cons}} = \Delta\mathrm{KE}$$

And then a little rearranging gives:

$$\Delta\mathrm{PE} + \Delta\mathrm{KE} \;=\; \Delta(\mathrm{PE} + \mathrm{KE}) \;=\; W_{\mathrm{non-cons}}$$

One more definition and we're done. We define the **total energy** E to be the sum of the potential and kinetic energies:

$$E = \mathrm{PE} + \mathrm{KE} \tag{6.10}$$

and then the last equation becomes:

$$\Delta E = \Delta(\mathrm{PE} + \mathrm{KE}) = W_{\mathrm{non-cons}} \tag{6.11}$$

In words, the change in the total energy of a system equals the work done by the non-conservative forces.

Oftentimes we have a situation where there are *no* non-conservative forces; for example if there are no friction force that need be considered. In that case, $W_{\mathrm{non-cons}}$ is zero and Eq. 6.11 reduces to

$$\Delta E = E_f - E_i = 0 \qquad \text{(No non-conservative forces!)} \tag{6.12}$$

which tells us that for this case the total energy *does not change*. When a quantity in physics remains the same in spite of other changes in the system we say that the quantity is **conserved**. So we would also say that in the absence of non-conservative forces, total energy is *conserved*.

6.1.8 Solving Problems With Energy Conservation

The principle of energy conservation can be useful in solving physics problems where the motion of a particle is complicated but the *kinds* of forces involved are simple, and also if you aren't being asked for a *time* interval—because time does not enter into our equations for energy.

In the cases where there are no friction forces (or any other forces which do work) we can use Eq. 6.12. If we setup expressions for the total energy of a system before and after some motion takes place (and set them equal) we may be able to solve for some unknown speed or distance.

If the problem does involve a friction force or some other force which does work we would have to use the more general Eq. 6.11 but it too can be useful if we have some way of treating the non–coservative force.

6.1.9 Power

One more concept involving work and energy is useful in studying the world. We've discussed how one calculates work done by a force. This quantity involved *force* and *distance* but it did not involve *time*. If an amount of work W is done in a time t we define the average power from this force as

$$\overline{P} = \frac{W}{t} \tag{6.13}$$

Power is the expression of how *rapidly* energy is transferred from one system to another.

Like work and energy, power is a *scalar* and its units are those of energy (joules) divided by those of time (seconds). We call the combination $\frac{\mathrm{J}}{\mathrm{s}}$ a **watt**:

$$1\,\mathrm{watt} = 1\,\mathrm{W} = 1\,\tfrac{\mathrm{J}}{\mathrm{s}} = 1\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s^3}}$$

If we consider very small time intervals t in Eq. 6.13 we find the *instantaneous* power P from the force.

In the special case of a particle moving in one dimension, acted on by a constant force along this direction and moving with instantaneous velocity v_x, we can find the instantaneous power from:

$$P = F_x v_x \tag{6.14}$$

6.2 Worked Examples

6.2.1 Kinetic Energy

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1. At what speed does a $1000\,\mathrm{kg}$ compact car have the same kinetic energy as a $20,000\,\mathrm{kg}$ truck going at $25\,\mathrm{km/hr}$? [KJF 10-8]

Let m and v be the mass and speed of the compact car and M and V be the mass and speed of the truck. We don't know v but we have the condition that the kinetic energies of the two are equal. Thus:

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}MV^2 \qquad \Longrightarrow \qquad v^2 = \frac{MV^2}{m} = \frac{(2.00\times 10^4\,\mathrm{kg})(25\,\frac{\mathrm{km}}{\mathrm{hr}})^2}{(1.00\times 10^3\,\mathrm{kg})}$$

Solving this for v gives

$$v = 112\,\tfrac{\mathrm{km}}{\mathrm{hr}}$$

6.2.2 The Spring Force

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2. An ideal spring has a force constant of $820\,\frac{\mathrm{N}}{\mathrm{m}}$. How far should one deform it from its equilibrium length so that $0.100\,\mathrm{J}$ of energy is stored?

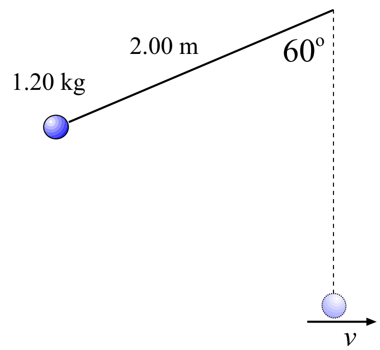

Figure 6.3: Mass on the end of a string is pulled back by 60° and then released. When the mass reaches the lowest point, what is the tension in the string?

Use Eq. 6.9 and solve for x:

$$\mathrm{PE_{spr}} = \tfrac{1}{2}kx^2 \qquad \Longrightarrow \qquad x^2 = \frac{2(\mathrm{PE_{spr}})}{k}$$

Plug in the numbers:

$$x^2 = \frac{2(0.100\,\mathrm{J})}{(820\,\frac{\mathrm{N}}{\mathrm{m}})} = 2.44 \times 10^{-4}\,\mathrm{m}^2 \qquad \Longrightarrow \qquad x = 1.6 \times 10^{-2}\,\mathrm{m} = 1.6\,\mathrm{cm}$$

We need to compress the spring by 1.6 cm.

6.2.3 Solving Problems With Energy Conservation

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3. A small $1.20\,\mathrm{kg}$ mass is attached to the end of a string of length $2.00\,\mathrm{m}$; the string is pulled back by 60.0° from the vertical, as shown in Fig. 6.3. The mass is released and it swings downward on the string. If the string breaks under a tension of more than $20\,\mathrm{N}$, will the mass be able to get to the bottom of its swing as shown in the figure?

For reasons that we'll see as we work the problem, the *maximum* tension in the string will occur at the bottom of the swing so if the string ever breaks it will break *then*. So we will assume it gets to the bottom and then calculate the string tension at that point.

The string tension at the bottom is *not* equal to the weight of the mass. We have to analyze the forces and apply Newton's 2nd law!

The forces acting on the mass at the bottom of the swing are shown in Fig. 6.4. They are gravity, mg downward and the string tension T upward. These forces do not add to zero

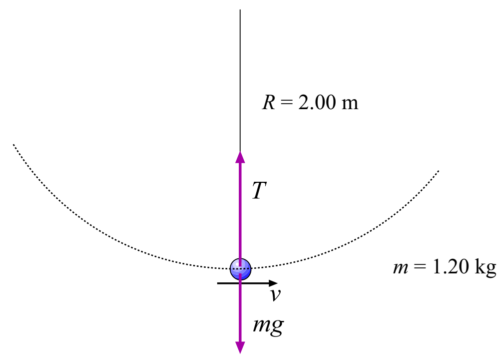

Figure 6.4: Forces acting on the mass at the bottom of the swing.

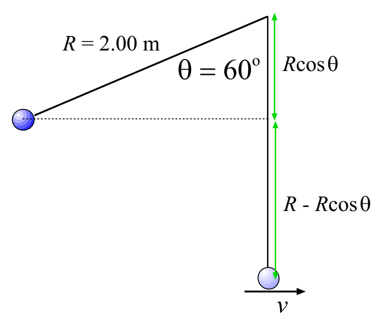

Figure 6.5: Some geometry for Example 3.

because at this point the mass *is* accelerating. How so? The mass is moving on the arc of a circle of radius R with speed v and so the radial (inward) forces must add to give the centripetal force, $F_c = mv^2/R$. Thus:

$$T - mg = \frac{mv^2}{R} \tag{6.15}$$

where R is the string length, $R = 2.0\,\mathrm{m}$. To find T we will need to know the speed v at the bottom of the swing.

We can get v using the conservation of energy. There are no friction forces here, only the conservative force of gravity and the string tension T which does no work because it always pulls perpendicularly to the motion of the mass. So total mechanical energy is conserved, $E_i = E_f$.

Taking the bottom of the swing to be "zero height" a little geometry (see Fig. 6.5) show that the initial height of the mass was

$$y_i = R - R\cos\theta = R(1-\cos\theta) = (2.00\,\mathrm{m})(1-\cos 60^\circ) = 1.00\,\mathrm{m}$$

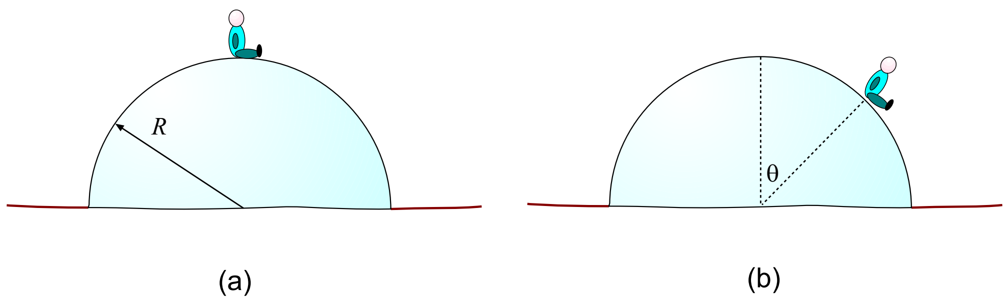

Figure 6.6: (a) Boy rests at top of frictionless hemisphere; he starts to slip down one side of it! (b) At some angle θ from the vertical he loses contact with the surface.

So energy conservation gives

$$\mathrm{PE}_i + \mathrm{KE}_i = \mathrm{PE}_f + \mathrm{KE}_f \qquad \Longrightarrow \qquad mgy_i + 0 = mgy_f + \tfrac{1}{2}mv^2$$

Here cancel a factor of m and do some algebra:

$$g(y_i - y_f) = \tfrac{1}{2}v^2 \qquad \Longrightarrow \qquad v^2 = 2g(y_i - y_f)$$

Plug in the numbers and get

$$v^2 = 2(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.00\,\mathrm{m} - 0) = 1.96\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2} \qquad \Longrightarrow \qquad v = 4.43\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Now we have the value of v^2 that we can use in Eq. 6.15. Some algebra on that equation gives

$$T = mg + \frac{mv^2}{R} = m\left(g + \frac{v^2}{R}\right)$$

Now plug in the numbers and get

$$T = (1.20\,\mathrm{kg})\left((9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) + \frac{(19.6\,\frac{\mathrm{m}^2}{\mathrm{s}^2})}{(2.00\,\mathrm{m})}\right) = 23.5\,\mathrm{N}$$

But the string cannot support a tension this large! So the string will break before the mass gets to the bottom of the swing.

---

4. A small boy sits at the top of a frictionless hemisphere of radius R, as shown in Fig. 6.6(a). He starts to slip down one side of it and at some angle θ measured from the vertical he loses contact with the surface, as shown in Fig. 6.6(b). Find the angle θ.

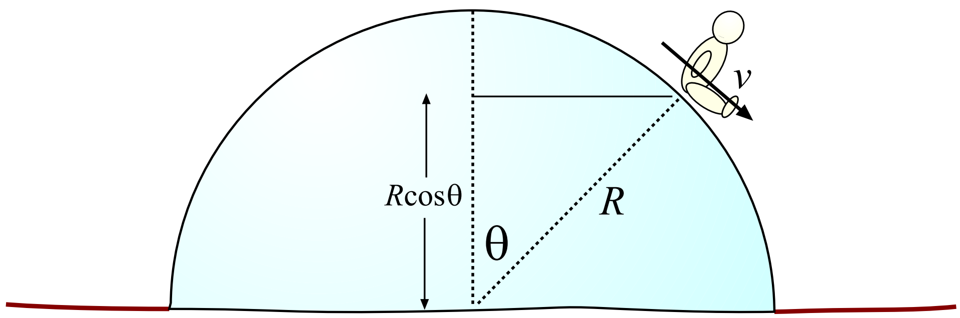

Figure 6.7: Boy at position given by θ has speed v and height $R\cos\theta$

This is a classic and somewhat challenging problem but it does not require any more math than simple trig.

We focus on the point at which the loss of contact occurs. While the boy starts from rest at the top of the sphere, his speed is v at this point. As mentioned, his position is given by the angle θ as measured form the vertical so that while his initial height was R it is now $R\cos\theta$; see Fig. 6.7.

Since there are no friction forces acting, the boy's total energy is conserved between the initial position at the top and the final position at θ, that is:

$$\mathrm{KE}_i + \mathrm{PE}_i = \mathrm{KE}_f + \mathrm{PE}_f$$

Using $\mathrm{PE} = mgy$ this is

$$0 + mgR = \tfrac{1}{2}mv^2 + mgR\cos\theta$$

where m is the mass of the boy. Do a little algebra on this and get:

$$\tfrac{1}{2}mv^2 = mgR - mgR\cos\theta = mgR(1-\cos\theta) \qquad \Longrightarrow \qquad v^2 = 2gR(1-\cos\theta)$$

The condition that the boy "loses contact" with the surface means that there is no *normal force* from the surface on the boy at some point. We know that the surface exerts no sideways (tangential) force on the boy because there is no friction. But in general the surface can push outward on the boy with some force F_N. (It cannot pull inward, and that is why the boy will fly off at some point.)

The forces on the boy when he is at position θ are shown in Fig. 6.8. Gravity mg pulls downward and the normal force F_N pushes outward. The force of gravity has been split into components: Doing a little geometry, we can see that a component $mg\cos\theta$ points inward toward the center of the sphere and a component $mg\sin\theta$ points tangentially.

Now at the moment the boy loses contact he is following the path of a circle (with speed v) so that the net force *in the inward direction* is the centripetal force, mv^2/R. (It is true

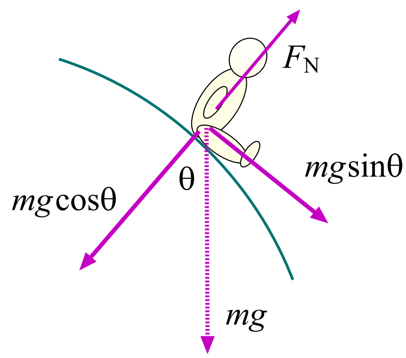

Figure 6.8: Forces on the boy at position given by θ.

that there is also a tangential force but that doesn't matter.) According to our force diagram the net force in the inward direction is

$$F_c = mg\cos\theta - F_N$$

and so

$$\frac{mv^2}{R} = mg\cos\theta - F_N\ .$$

But at the moment he loses contact, $F_N = 0$ so we get

$$\frac{mv^2}{R} = mg\cos\theta \qquad \Longrightarrow v^2 = Rg\cos\theta$$

Comparing our results for energy conservation and the centripetal force, we got two expressions for v^2. If we equate them, we get

$$2gR(1-\cos\theta) = Rg\cos\theta$$

Cancel the Rg and do some algebra:

$$2(1-\cos\theta) = \cos\theta \qquad \Longrightarrow \qquad 2 = 3\cos\theta$$

Finally,

$$\cos\theta = \frac{2}{3} \qquad \Longrightarrow \qquad \theta = 48.2^\circ$$

Chapter 7

Momentum

7.1 The Important Stuff

7.1.1 Momentum; Systems of Particles

Again we begin the chapter with a definition; later, we'll see why it's useful and then use a new principle to solve problems in physics.

When a particle of mass m has velocity $\mathbf{v}$, its **momentum** (or more specifically, its **linear momentum**), $\mathbf{p}$ is defined as:

$$\mathbf{p} = m\mathbf{v} \tag{7.1}$$

That is, momentum is just mass times velocity.

Momentum is a *vector*, and Eq. 7.1 means that

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z$$

The units of momentum are those of mass (kg) times those of velocity ($\frac{\mathrm{m}}{\mathrm{s}}$), that is, they are $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$. Oddly enough people haven't been able to agree on a special name for this combination so we leave it as it is.

7.1.2 Relation to Force; Impulse

Using Newton's 2nd law we see how momentum is related to force. Considering forces and motion in one dimension, if a constant force F_x acts on a mass m for a time interval Δt then

$$F_x \Delta t \,=\, (ma_x)\Delta t \,=\, m(a_x \Delta t) \,=\, m\Delta v_x$$

But since

$$m\Delta v_x = m(v_{\mathrm{f}x} - v_{\mathrm{i}x}) \,=\, mv_{\mathrm{f}x} - mv_{\mathrm{i}x} = \Delta p_x$$

then combining the two equations gives

$$F_x \Delta t = \Delta p_x$$

Or, dividing by Δt,

$$F_x = \frac{\Delta p_x}{\Delta t} \tag{7.2}$$

so that the force on a particle is equal to the rate of change of its momentum.

Eq. 7.2 gives us a definition for the **average force** acting on a particle. If, during a time interval Δt the change in momentum of a particle is $\Delta \mathbf{p}$, then the average force on the particle for that period is

$$\overline{\mathbf{F}} = \frac{\Delta \mathbf{p}}{\Delta t} \tag{7.3}$$

People often make a definition to deal with the change in momentum of a particle. If over a certain time interval a particle has a change in momentum $\Delta \mathbf{p}$, they say that the **impulse**, $\mathbf{I}$ imparted to the particle is

$$\mathbf{I} = \Delta \mathbf{p} \tag{7.4}$$

From this, we see that impulse is a vector with the same units as momentum.

Since from Eq. 7.3 we have $\overline{\mathbf{F}} \Delta t = \Delta \mathbf{p}$, from the definition of impulse we also have

$$\overline{\mathbf{F}} \Delta t = \mathbf{I}$$

7.1.3 The Principle of Momentum Conservation

Oftentimes in physics we are faced with the problem of two objects which are moving along free from forces; the objects interact with *each other* and then move apart, again free from outside forces, as shown in Fig.7.1. Often we want to call such an event a "collision". The velocities (both magnitude and direction) of the masses after the event can be quite different from what they were before the event.

The problem with the analysis of such an event is that the force *between the two objects* may not be known very well. Generally when objects collide this force is very strong, very brief and very complicated. In such a situation does physics have anything at all to say about the collision?

Here I'd like to give a simple derivation of the principle of momentum conservation because the math is simple and it's important to understand the basic reason behind this principle.

Consider what happens during the interaction between A and B in Fig. 7.1(b). They exert forces on each other, but from Newton's 3rd law these forces are equal and opposite:

$$\mathbf{F}_{\mathrm{B\,on\,A}} = -\mathbf{F}_{\mathrm{A\,on\,B}} \tag{7.5}$$

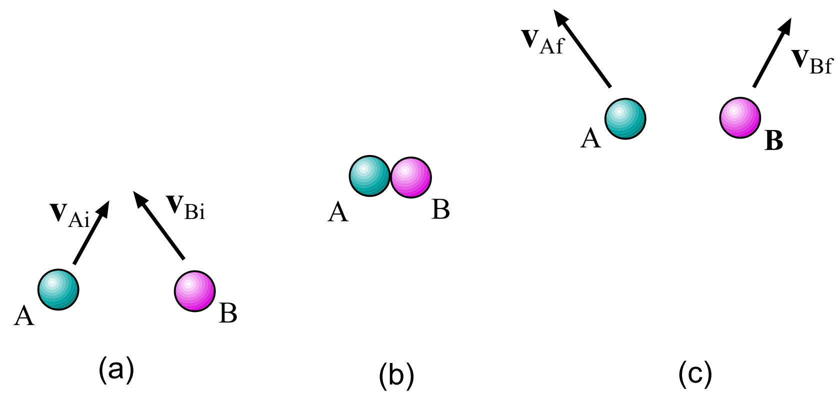

Figure 7.1: A collision between masses A and B. (a) Masses move freely with velocities $\mathbf{v}_{Ai}$ and $\mathbf{v}_{Bi}$. (b) For a brief time the masses exert a force on one another. The force changes their velocities. (c) The masses move freely again, with velocities $\mathbf{v}_{Af}$ and $\mathbf{v}_{Bf}$

Consider a very small time interval Δt during the interaction over which we can take force as being constant. Multiply both sides of Eq. 7.5 by Δt and get:

$$(\mathbf{F}_{\rm B\,on\,A})\Delta t = -(\mathbf{F}_{\rm A\,on\,B})\Delta t \tag{7.6}$$

Now we're assuming that B is the *only* thing exerting a force on A and vice versa. That means that $\mathbf{F}_{\rm B\,on\,A}\Delta t$ gives the change in momentum of A, $\Delta\mathbf{p}_A$ over this time interval Δt . Similar remarks hold for B so we can write:

$$(\Delta\mathbf{p}_A)_{\rm interval\,\Delta t} = -(\Delta\mathbf{p}_B)_{\rm interval\,\Delta t} \tag{7.7}$$

So the *changes in momentum* for A and B are equal and opposite over every little bit of the interaction. If we add up the changes in momentum over all little bits of the interaction we still find that they are equal and opposite for the *whole* interaction period :

$$(\Delta\mathbf{p}_A)_{\rm whole} = -(\Delta\mathbf{p}_B)_{\rm whole} \tag{7.8}$$

Now write the changes in momentum in terms of the momenta before and after the interaction:

$$\mathbf{p}_{Af} - \mathbf{p}_{Ai} = -\left(\mathbf{p}_{Bf} - \mathbf{p}_{Bi}\right) \tag{7.9}$$

A little rearranging gives:

$$\mathbf{p}_{Ai} + \mathbf{p}_{Bi} = (\mathbf{p}_{Af} + \mathbf{p}_{Bf}) \tag{7.10}$$

Now a definition. We'll let $\mathbf{P}$ stand for the **total momentum** of all the particles,

$$\mathbf{P} = \mathbf{p}_A + \mathbf{p}_B \tag{7.11}$$

With this definition, then the left side of 7.10 is the initial value of **P** and the right side is the final value of **P**. So we have:

$$\mathbf{P}_i = \mathbf{P}_f \tag{7.12}$$

that is, the value of the total momentum *stays the same*, or as we say in physics, is *conserved*. We need to recall the conditions under which we could make this statement (our assumptions): We assumed that during the interaction, A and B were interacting with *each other* but felt no forces from anything else. Another way to say this is that during the interaction, the two objects were **isolated**; because of that, their total momentum was conserved.

The principle can be made more general by including more particles, and the lesson can be stated as:

For a system of isolated particles, the total momentum is conserved.

7.1.4 Collisions; Problems Using the Conservation of Momentum

Sometimes we are faced with a problem where there are two or interacting particles which "feel" no forces from anything else, i.e. they are isolated. Sometimes we have a situation where we are considering the motion of several particles over such a *short period of time* that we can safely ignore the external forces but we can't ignore the forces between the particles. Then, for the purposes of the problem, the particles are (again) "isolated". The forces between the particles may be very complicated, but that doesn't matter. The total momentum of the particles will stay the same before and after the interaction.

Examples of where such an interaction can occur are:

• When two particles travel freely, bounce off one another and then travel away from one another in new directions; this is what we normally think of as a "collision".
• When two particles come together and stick to one another; the combined mass travels off as one unit (with a mass equal to the sum of the individual masses).
• When a single mass explodes and the individual parts fly off in different directions; we would call this an "explosion". Again, the sum of all the masses is the same before and after.

Though total *momentum* stays the same during these processes, what about the *energy* of the particles? More specifically, since in the rapid interactions we are considering the potential energy of the system doesn't change by much we ask what happens to the *kinetic energy* of the particles.

In general, the total kinetic energy can remain the same or decrease or even increase for a collision; it depends on the nature of the interacting objects and the kind of force the exert on each other. The usual case is that when real objects collide the force is in small part frictional in nature and then kinetic energy is *lost*, or rather it changes form to *thermal* energy. But it is possible that the impact releases chemical or other stored energy from an exploding element and then the kinetic energy would *increase*. But if the surfaces

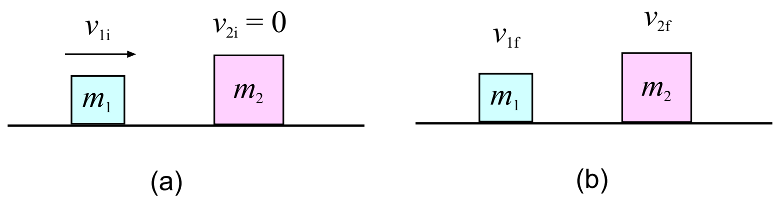

Figure 7.2: Elastic collision in one dimension. (a) Mass m_1 has initial velocity v_{1i} and mass m_2 is at rest. (b) Final velocities of the two masses are v_{1f} and v_{2f}. Formulae for these velocities are given in the text.

of the objects are very tough and springy the change in energy might be so small as to be unmeasurable.

When the total kinetic energy stays the same in a collision we say that it is an **elastic** collision. If KE is lost (or gained) we say that the collision is **inelastic**.

A special name is given to the case mentioned above where one particle strikes another and sticks. Such a collision is called **completely inelastic**.

If we know beforehand that a collision is elastic then we have some additional information which can help us solve a problem. As an example we consider a situation which is easy to set up in the lab: A one-dimensional collision in which a mass m_1 is moving toward another mass m_2 which is at rest. *Knowing that the collision is elastic* we want to find the velocities of the masses after the collision.

The problem is diagrammed in Fig.7.2. In the collision there are no (appreciable) external forces, so momentum is conserved:

$$m_1 v_{1i} + 0 = m_1 v_{1f} + m_2 v_{2f} \tag{7.13}$$

Here the v's are *velocities* so that they can be positive or negative. However if we say that v_{1i} is positive then v_{2f} had better be positive; if the second mass went backward after the collision, *both* masses would have to be moving backward, which is nonsense! But v_{1f} could possibly be negative; that is the case where m_1 bounces backwards after the collision.

Now if we *also* know that kinetic energy is conserved in the collision then we can write another equation:

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \tag{7.14}$$

If we take the masses m_1 and m_2 and v_{1i} as "known" and the final velocities v_{1f} and v_{2f} as unknown then the two equations 7.13 and 7.14 allow us to find the two unknown velocities. This involves some algebra which I'll skip, but the solution is:

$$v_{1f} = \frac{(m_1 - m_2) v_{1i}}{(m_1 + m_2)} \qquad\qquad v_{2f} = \frac{2 m_1 v_{1i}}{(m_1 + m_2)} \tag{7.15}$$

Eq. 7.15 tells us some interesting things. If the masses are equal ($m_1 = m_2$) then the equations tell us that v_{1f} is *zero*— the first mass stops— and $v_{2f} = v_{1i}$, the second mass moves off with the same velocity that the first mass had.

If we consider the case where m_2 is enormous compared to m_1, then in the solution for v_{1f} we can replace $(m_1 - m_2)$ by $-m_2$ and $(m_1 + m_2)$ by m_2, giving

$$v_{1f} = \frac{-m_2}{m_2} v_{1i} = -v_{1i}$$

so that mass m_1 just reverses its motion. In this case, m_2 will move forward very slowly.

The case where m_1 is enormous compared to m_2 is interesting. We find that v_{1f} is almost the same as v_{1i}, but for m_2 we find that since $(m_1 + m_2)$ is basically the same as m_1 we get

$$v_{2f} \approx \frac{2m_1}{m_1} v_{1i} = 2v_{1i}$$

that is, m_2 goes forward with *twice* the original speed of m_1. (And m_1 plows ahead with roughly the same speed.)

7.1.5 Systems of Particles; The Center of Mass

Newton's laws as given in Chapter 4 really apply to point masses ("particles") and yet we've been applying them to real objects which have non–zero dimensions. Have we been correct in doing so? If so, *how* was it correct?

There are two problems in ignoring the fact that real objects have real sizes. First off, a real object can change its orientation while staying basically in the same place. We say that a real object can *rotate*, and we will deal with simple rotations in the next chapter.

Secondly, if an object has a non-zero size it is somewhat problematical as to what we mean by the "location" of the object when discussing its kinematics: When we talk about the "motion" of the object, what particular point of the object are we talking about? The "middle" of the object, maybe? What do we mean by "middle"?

A real object can be treated as a collection of a whole lot of point particles and with this treatment and the use of Newton's laws (which do apply to points) one can show that we can *keep* using Newton's laws for big things (or systems of particles) as long as we make the following replacements:

• For the force $\mathbf{F}_{\rm net}$ in Newton's 2nd law, use the total *external* force acting on all parts of the object. (The parts of the object or set of masses may be exerting forces on *each other*, but we don't need to count those.)
• For the mass m, use the total mass of the object or the total mass of the system of particles. Let's call the total mass M.
• For the acceleration $\mathbf{a}$, realize that we are now talking about the acceleration of a *special point* called the **center of mass** of the object or system of particles: $\mathbf{a}_{\rm CM}$.

With these replacements in mind, the expression of Newton's 2nd law (fortunately) looks the same:

$$\mathbf{F}_{\rm net} = M\mathbf{a}_{\rm CM} \tag{7.16}$$

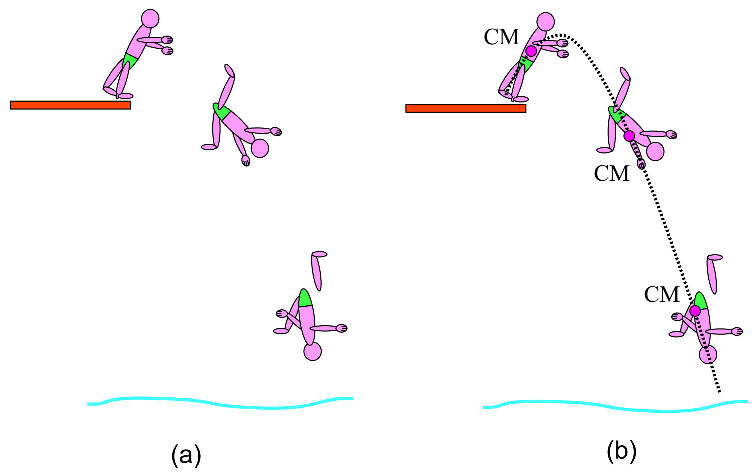

Figure 7.3: (a) Diver jumps off diving board and does some of that crazy diver-type stuff; action looks very complicated! (b) If we plot the center of mass of the diver, its motion is fairly simple (a parabolic path).

but the terms have new meanings.

With the new usage of Newton's second law, things which seem to have a complicated motion have a simple aspect. Consider a diver jumping into the water and doing all those crazy things that divers do, as shown in Fig.7.3. The motion looks complicated, but if at each instant in time we could locate the center of mass of the diver and track *its* motion, we would see that it follows a nice parabola as shown in Fig. 7.3(b). Why is that?

The revised version of Newton's 2nd law says to consider the total (external) force acting on the diver. Granted, gravity acts on the diver's various parts but the *total* force is still Mg, directed downward. The total mass of the diver is M. So we know that the acceleration of the *center of mass* is simple; it has magnitude $Mg/M = g$ and is directed downward, the same as for the (point) projectile problem solved back in Chapter 3. There we found that the general path is a parabola and so that is the shape of the path of the center of mass for the diver.

7.1.6 Finding the Center of Mass

So the center of mass has great importance in the physics of a system of particles. How do we find it?

Here we will just give the formula the center of mass of a set of mass points; for a continuous mass we use the same principle but it becomes necessary to use calculus. Also we'll just work in two dimensions.

If a set of mass points with masses m_1, m_2, m_3... have coordinates (x_1, y_1), (x_2, y_2),

$(x_3, y_3)\ldots$ then the x coordinate of its center of mass is given by

$$x_{\rm cm} = \frac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_ix_i}{M} \qquad \text{with} \quad M = \sum_i m_i \tag{7.17}$$

Likewise the y coordinate of the center of mass is

$$y_{\rm cm} = \frac{m_1y_1 + m_2y_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_iy_i}{M} \qquad \text{with} \quad M = \sum_i m_i \tag{7.18}$$

In both cases what we are doing is taking a *weighted average* of the coordinates of the mass points, where the "weighting" is done with the *masses* of the points.

As the members of a set of mass points move around, the location of their center of mass will also move, and it will have its own velocity. The components of the velocity of the center of mass satisfy an equation similar to that of the *location* of the cm:

$$v_{{\rm cm},x} = \frac{m_1v_{1,x} + m_2v_{2,x} + \cdots}{m_1 + m_2 + \cdots} = \frac{P_x}{M} \qquad v_{{\rm cm},y} = \frac{m_1v_{1,y} + m_2v_{2,y} + \cdots}{m_1 + m_2 + \cdots} = \frac{P_y}{M} \tag{7.19}$$

or:

$$\mathbf{v}_{\rm cm} = \frac{\mathbf{P}}{M} \qquad \Longrightarrow \qquad \mathbf{P} = M\mathbf{v}_{\rm cm}$$

...which looks like the definition of momentum but here it gives the total momentum of a system of particles.

Finally, the acceleration of the center of mass satisfies

$$a_{{\rm cm},x} = \frac{m_1a_{1,x} + m_2a_{2,x} + \cdots}{m_1 + m_2 + \cdots} \qquad\qquad a_{{\rm cm},y} = \frac{m_1a_{1,y} + m_2a_{2,y} + \cdots}{m_1 + m_2 + \cdots} \quad . \tag{7.20}$$

7.2 Worked Examples

Chapter 8

Rotational Kinematics

8.1 The Important Stuff

8.1.1 Rigid Bodies; Rotating Objects

So far we have had much to say about the motion of objects and the way it is determined by the forces acting on those objects. But in all of our examples we were treating the objects as points (i.e. "particles"), assuming that if they did have any size or internal motion it was unimportant.

We will now deal with the fact that real objects have a non-zero size and it addition to the motion of their center of mass (the translational motion) they can also be in rotation.

We will only deal with objects which keep the same shape; the individual bits of the object always stay at the same distance from one another, i.e. they don't stretch, compress or otherwise deform. Such an object is called a **rigid body** and we'll have enough on our plate just to deal with such objects.

Even after restricting ourselves to rigid bodies, their motion can be very complicated. If you throw a football the right way it will have a simple kind of rotational motion, as shown in Fig. 8.1(a). If you throw it the wrong way it will be rotating in several different ways all at once, as in Fig. 8.1(b).

In this and the next chapter we will deal with very simple rotations, usually of the kind illustrated in Fig. 8.2(a). Here, a rigid body rotates around a fixed axis. Each point of the object moves in a circle, though the radii of these individual circles will not be the same. For now, this is what we will mean by a "rotating object".

We'll also consider rotations of the kind illustrated in Fig. 8.2(b). Here, a round object is *rolling*. We can look at this as a rotation; here the axis keeps the same orientation, though it is moving. This kind of motion will be covered in the following chapter.

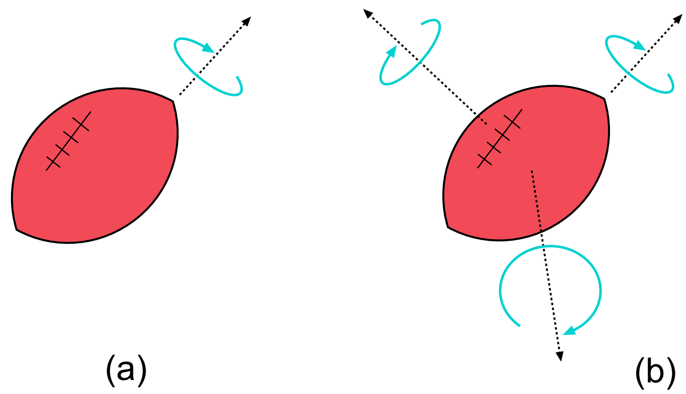

Figure 8.1: (a) Football thrown the right way. (b) Football thrown the way I always throw one. It has several kinds of rotation all at once!

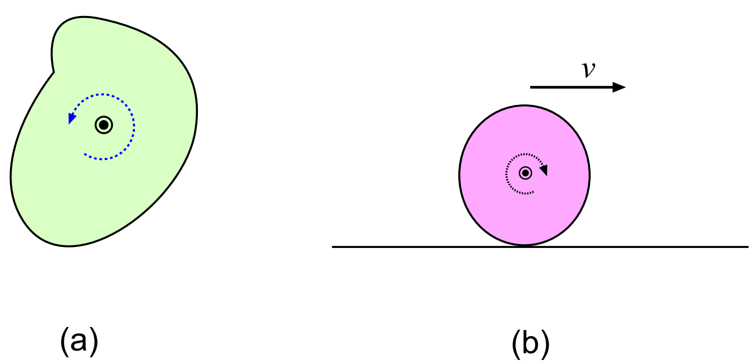

Figure 8.2: The kinds of (simple) rotations we will consider. (a) Object turns about a fixed axis. (b) Rolling object turns about an axis which itself is in motion.

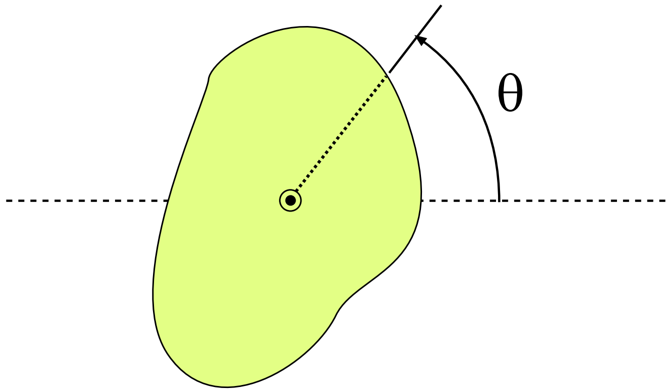

Figure 8.3: Orientation of a rotating object is given by the angle θ.

8.1.2 Angular Displacement

The orientation of a rotating object is given by a single number, an angle θ, which can be taken as the angle between some some radial marker on the object and some fixed axis, as shown in Fig. 8.3.

Now, while we have so far measured angles in *degrees* (since they are easiest to visualize) it turns out that it is now more convenient to measure angles in **radians**. Recall that there are 2π radians in a full circle, i.e. 360 degrees:

$$2\pi\,\text{radians} = 360\,\text{degreees} \qquad \text{or} \qquad \pi\,\text{radians} = 180\,\text{degrees} \tag{8.1}$$

Also keep in mind that 1 **revolution** just means that the object makes one full turn. To convert between the different "units" for rotations, use

$$1\,\text{revolution} = 360\,\text{degrees} = 2\pi\,\text{radians}$$

The reason for using radians to measure angles is that it will be useful to talk about the actual distance a point on a rotating object travels when the object rotates by an angle θ. As shown in Fig. 8.4, a point at distance r from the axis travels a distance s when the object rotates through an angle θ. There is a simple relation between these values: $s = r\theta$, which is true *only* when θ is measured in radians. Thus:

$$s = r\theta \qquad \text{With } \theta \text{ in radians!} \tag{8.2}$$

We will also refer to s as the "linear distance" through which the point travels, though of course the actual path is the arc of a circle.

When an object rotates all points have the same angular displacement, but since the points are at different r's, the linear distances they travel are different.

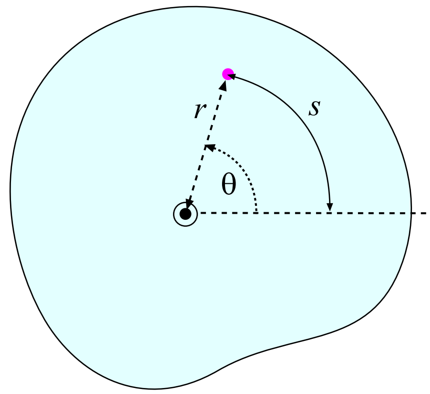

Figure 8.4: Object turns through an angle θ; a point on the object at a distance r from the axis moves through a distance $s = r\theta$.

8.1.3 Angular Velocity

We follow the same steps as in Chapter 2 to study the relation between angular displacement and time.

If a rotating object turns through some angular displacement $\Delta\theta$ in a time Δt then the **average angular velocity** $\overline{\omega}$ for that period is

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} \tag{8.3}$$

(The symbol ω is the small Greek letter "omega".)

As with our linear motion, a much more interesting quantity is the **instantaneous angular velocity** ω, which has the same kind of definition but applied to a very small time interval Δt and which than has meaning at a *particular time* t:

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad \text{For very small } \Delta t \tag{8.4}$$

As used in these notes, angular velocity is a scalar (a single number) because of the simplicity of our rotations; position is given by a single angle θ. In advanced physics courses where the rotations are more complicated it is necessary to treat angular velocity as a vector.

Since θ is measured in radians and t in seconds, the units of ω ought to be $\frac{\text{rad}}{\text{s}}$ and indeed if you are asked for an angular velocity you should give it with these units. But as we'll see if we stick by our practice of writing down *all* the units (including "rad") we will run into some inconsistencies later on, which basically come from the fact that our "radian" units is *mathematical* in nature; there is no standard "radian" kept under glass in France anywhere.

Though opinions differ on this, my practice is that the symbol "rad" should taken as *optional*, inserted just for clarity. When expressing an angular velocity or acceleration, one *should* include it to emphasize that we are *not* using degrees. When we do a calculation where the answer needs to come out in joules, then it should be dropped. However it would be OK (with me) to express an angular velocity in $\frac{1}{\mathrm{s}} = \mathrm{s}^{-1}$.

When we set up our equations for angular displacement it will be simplest to say that at time $t = 0$ the angular displacement is $\theta = 0$.

8.1.4 Angular Acceleration

As we'll see, when an outside influence acts on a rotating object the effect is that the rotational motion (i.e. the angular velocity) *changes* and we now want a measure of how rapidly the angular velocity changes with time. If the angular velocity of object has a change $\Delta\omega$ in a time period Δt We define an **average angular acceleration** $\overline{\alpha}$ as:

$$\overline{\alpha} = \frac{\Delta\omega}{\Delta t} \tag{8.5}$$

But a more important quantity is the **instantaneous angular acceleration**, defined by

$$\alpha = \frac{\Delta\omega}{\Delta t} \qquad \text{for very small } \Delta t \tag{8.6}$$

The units of angular acceleration must be the units of angular velocity divided by those of time, that is,

$$\frac{\frac{\mathrm{rad}}{\mathrm{s}}}{\mathrm{s}} = \frac{\mathrm{rad}}{\mathrm{s}^2} \quad .$$

But as with angular velocity, I wouldn't get upset if you expressed it as $\frac{1}{\mathrm{s}^2}$, treating the "rad" as optional. Check with the local Units Police officer.

8.1.5 The Case of Constant Angular Acceleration

If α does not depend on time then for any time interval t (where we start counting time from $t = 0$) we have

$$\alpha = \frac{\Delta\omega}{t} = \frac{\omega - \omega_0}{t} \quad .$$

Here, ω_0 is the initial angular velocity of the object, that is, its angular velocity at $t = 0$. This gives

$$\omega = \omega_0 + \alpha t \tag{8.7}$$

We followed very similar steps in finding the equation $v = v_0 + at$ in one-dimensional motion with constant *linear* acceleration. And in a similar way one can show that θ is given by

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 \tag{8.8}$$

Here we measure θ such that $\theta = 0$ at $t = 0$.

We can get other relations between θ, ω and α with some algebra. One can show that Eqs. 8.7 and 8.8 give:

$$\omega^2 = \omega_0^2 + 2\alpha\theta \tag{8.9}$$

And one can show

$$\theta = \tfrac{1}{2}(\omega_0 + \omega)t \tag{8.10}$$

but you are cautioned that *this* equation holds only when we already know that α is constant.

8.1.6 Relation Between Angular and Linear Quantities

We've already mentioned that when the object turns by an angle θ a particular point on the object at radius r travels a distance s, given by $s = r\theta$.

Suppose an object is rotating with angular velocity ω. A point at radius r has speed v which we can find from

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega$$

(Here we are really talking about a very small time interval t so that v and ω are the *instantaneous* velocities.)

This gives us the relation between *linear* speed v and *angular* speed ω for the point:

$$v = r\omega \tag{8.11}$$

We note that while all points of the object rotate at the same angular speed ω, because they have different radii they will have different *linear* speeds v.

Finally we discuss the acceleration of a point on the rotating object. This is little more complicated, because there are two parts to the acceleration.

We have already seen that when a particle undergoes *uniform* circular motion its acceleration points toward the center of the circle and has magnitude $a_c = \frac{v^2}{r}$. We can use Eq. 8.11 to express this as:

$$a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \tag{8.12}$$

But when the object has an angular acceleration there is also a component of the acceleration in the *tangential* direction. This is because with an α which is not zero, the linear speed of the point is increasing as it travels on its circular path. Again taking a very small time interval t, the tangential acceleration is given by

$$a_T = \frac{\Delta v}{t} = \frac{\Delta(r\omega)}{t} = r\frac{\Delta\omega}{t} = r\alpha$$

That is,

$$a_T = r\alpha \tag{8.13}$$

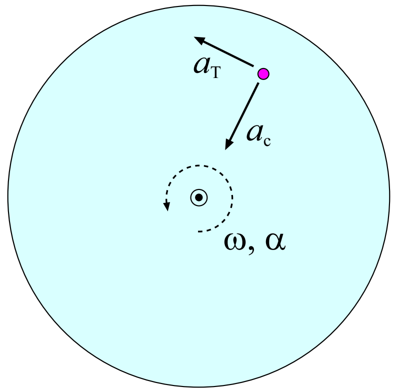

Figure 8.5: Acceleration of a point on a rotating object has centripetal and tangential components, a_c and a_T.

So in general for a point on a rotating object, the acceleration has *two* components, as illustrated in Fig. 8.5.

8.2 Worked Examples

8.2.1 Angular Displacement

1. A rigid body turns through 1.85 radians. Express this in degrees and revolutions

Use the fact that π radians equals 180°, then

$$1.85\,\mathrm{rad} \;=\; (1.85\,\mathrm{rad})\left(\frac{180\,\mathrm{deg}}{\pi\,\mathrm{rad}}\right) = 106\,\mathrm{deg}$$

Then, use the fact that 1 revolution $= 2\pi rad$ to get

$$1.85\,\mathrm{rad} \;=\; (1.85\,\mathrm{rad})\left(\frac{1\,\mathrm{rev}}{2\pi\,\mathrm{rad}}\right) = 0.294\,\mathrm{rev}$$

8.2.2 Angular Velocity and Acceleration

2. Long ago people listened to music which was stored on "phonograph records". These records turned at a rate of 33.3 revolutions per minute. Express this rotation rate in radians per second.

Use the relations $1\,\text{rev} = 2\pi\,\text{rad}$ as well as $1\,\text{min} = 60\,\text{s}$ to convert the units:

$$33.3\,\tfrac{\text{rev}}{\text{min}} = (33.3\,\tfrac{\text{rev}}{\text{min}})\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 3.49\,\tfrac{\text{rad}}{\text{s}}$$

8.2.3 Rotational Motion with Constant Angular Acceleration

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3. A flywheel has a constant angular deceleration of $2.0\,\frac{\text{rad}}{\text{s}^2}$. (a) Find the angle through which the flywheel turns as it somes to rest from an angular speed of $220\,\frac{\text{rad}}{\text{s}}$. (b) Find the time required for the flywheel to come to rest. [CJ7 8-19]

(a) The problem tells us that if we take the angular velocities as positive, then we have $\alpha = -2.0\,\frac{\text{rad}}{\text{s}^2}$. Then if the initial angular velocity is $\omega_0 = 220\,\frac{\text{rad}}{\text{s}}$ and it comes to rest ($\omega = 0$) as it turns through an angle θ, the relation between these quantities is given by Eq. 8.9,

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

so solving for θ,

$$\theta = \frac{(\omega^2 - \omega_0^2)}{2\alpha}$$

Plugging in the numbers,

$$\theta = \frac{(0)^2 - (220\,\frac{\text{rad}}{\text{s}})^2}{2(-2.0\,\frac{\text{rad}}{\text{s}^2})} = 1.21 \times 10^4\,\text{rad}$$

(b) To find the time it takes the flywheel to come to rest, use Eq. 8.7,

$$\omega = \omega_0 + \alpha t \qquad \Longrightarrow \qquad t = \frac{(\omega - \omega_0)}{\alpha}$$

and we get

$$t = \frac{(0 - 220\,\frac{\text{rad}}{\text{s}})}{(-2.0\,\frac{\text{rad}}{\text{s}^2})} = 110\,\text{s}$$

8.2.4 Relation Between Angular and Linear Quantities

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4. A string trimmer is a tool for cutting grass and weeds. It utilizes a length of nylon "string" that rotates about an axis perpendicular to one end of the string.

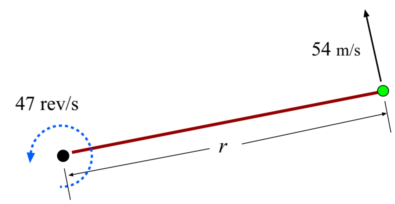

Figure 8.6: Rotating string in Example 4.

The string rotates at an angular speed of $47\,\frac{\text{rev}}{\text{s}}$, and its tip has a tangential speed of $54\,\frac{\text{m}}{\text{s}}$. What is the length of the rotating string? [CJ7 8-29]

The rotating string is shown in Fig. 8.6. The *angular* frequency of the string's rotation is

$$\omega = 2\pi f = 2\pi(47\,\text{s}^{-1}) = 295\,\text{s}^{-1}$$

We have the speed of the tip of the string (which is a distance r from the axis), so from Eq. 8.11 we can get the length of the string, r:

$$v = r\omega \qquad \Longrightarrow \qquad r = \frac{v}{\omega} = \frac{(54\,\frac{\text{m}}{\text{s}})}{295\,\text{s}^{-1})} = 0.183\,\text{m} = 18.3\,\text{cm}$$

The string is 18.3 cm long.

Chapter 9

Rotational Dynamics

9.1 The Important Stuff

9.1.1 Introduction

Having worked with the *kinematics* of rotation in the last chapter we now move on to study the *dynamics* of rotation. In effect, we have to re-do that past chapters on dynamics (force, energy and momentum) in the setting of *rotating* objects. We have our work cut out for us and accordingly, this is a long chapter!

9.1.2 Rotational Kinetic Energy

When an object turns around a fixed axis (as they did in the last chapter) the individual bits are in motion, so the object certainly has *kinetic energy*. Kinetic energy is always a positive scalar , so the kinetic energies of these little bits add up to give the kinetic energy of the whole object.

We would like to calculate the kinetic energy, but we must keep in mind that even though all parts of the object have the same *angular velocity* they have different *speeds* because they lie at different distances from the axis. We will have to add up the kinetic energies of the separate parts of the object; this is another derivation for which it is instructive (and easy) to understand all the steps.

Imagine the object is broken up into little pieces, each indexed by the number i. Piece i has mass m_i, speed v_i and sits at a distance r_i from the axis, as shown in Fig.9.1.

The total kinetic energy of rotation is the sum of the kinetic energies of all the little bits:

$$\mathrm{KE}_{\mathrm{rot}} = \sum_i \tfrac{1}{2} m_i v_i^2$$

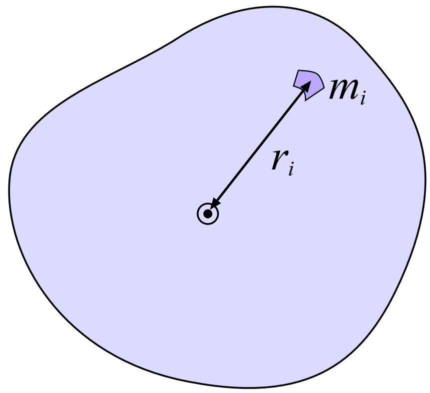

Figure 9.1: A little piece of the rotating object has mass m_i and is a distance r_i from the axis.

Into this we substitute $v_i = r_i\omega$, giving

$$\mathrm{KE_{rot}} = \sum_i \tfrac{1}{2} m_i (r_i\omega)^2$$

Now we take the common factors of $\frac{1}{2}$ and ω^2 outside the sum and write it as

$$\mathrm{KE_{rot}} = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \tag{9.1}$$

and now we focus on the quantity inside the parentheses in Eq. 9.1.

This quantity is the sum over all the little pieces of the object of the *mass* of that piece times its radius squared. (When the object is a continuous mass like a disc it is implied that the pieces need to be very small!) This quantity is called the **moment of inertia** of the object[1], and it is given the symbol I. Thus:

$$I = \sum_i m_i r_i^2 \tag{9.2}$$

As we will use it, the moment of inertia is a scalar. It units must be those of mass times those of length squared, thus for physics they are

$$\mathrm{kg}\cdot\mathrm{m}^2$$

There is no abbreviation for this.

Using this definition in Eq. 9.1 we have the simple expression

$$\mathrm{KE_{rot}} = \tfrac{1}{2} I\omega^2 \tag{9.3}$$

and we note that it looks like the expression for the kinetic energy of a particle, $\mathrm{KE} = \frac{1}{2}mv^2$.

[1]Sometimes you will see it called the **rotational inertia**.

9.1.3 More on the Moment of Inertia

Most of the time we are interested in the moment of inertia of some continuous object like a rod or a disk, and for these shapes, even though we use the *principle* given in Eq. 9.2 we actually need calculus to get the result.

So it is best just to give a short catalogue of formulae to use with the most common basic shapes we'll see in our problems.

Figures 9.2 and 9.3 give a set for formulae for the moments of inertia for various shapes. In all these cases we assume that the material of the object has uniform density. To use the formulae, we need to know the mass of the object and one or two of its dimensions. Some of the most commonly used formulae are:

$$I_{\rm hoop} = MR^2 \qquad I_{\rm disk} = \tfrac{1}{2}MR^2 \qquad I_{\rm solid\ sph} = \tfrac{2}{5}MR^2 \tag{9.4}$$

It's important to realize that the value of the moment of inertia *depends on where we put the axis*. For example, for a uniform rod, the moment of inertia when the axis goes through the *middle* is $\frac{1}{12}ML^2$. But when the axis goes through the *end* of the rod, the moment of inertia is $\frac{1}{3}ML^2$.

There is a useful theorem which can give us the moment of inertia under a special circumstance. Suppose:

• We know that moment of inertia of some object around an axis which goes through its center of mass, $I_{\rm cm}$.

• We *want* the moment of inertia around an axis which is parallel to that axis, a distance D away from it.

Then we can get the desired moment of inertia from a simple formula,

$$I = I_{\rm cm} + MD^2 \quad , \tag{9.5}$$

a formula which is called the **Parallel Axis Theorem**. The elements of the theorem are illustrated in Fig. 9.4.

9.1.4 Torque

When we arrived at Chapter 4 we asked the question: "Accelerations are what makes motion interesting... what *causes* an object to accelerate?". We do something analogous here: "*Angular* accelerations are the interesting feature of rotational motion; what *makes* a rotating object accelerate?"

Actually, forces are still the cause for changes in motion, but for our present purposes the best answer is that changes in rotational motion are caused by a rotational version of force called **torque**. Torque has something to do with the force exerted on the object but it also depends on where the forces are exerted and their directions.

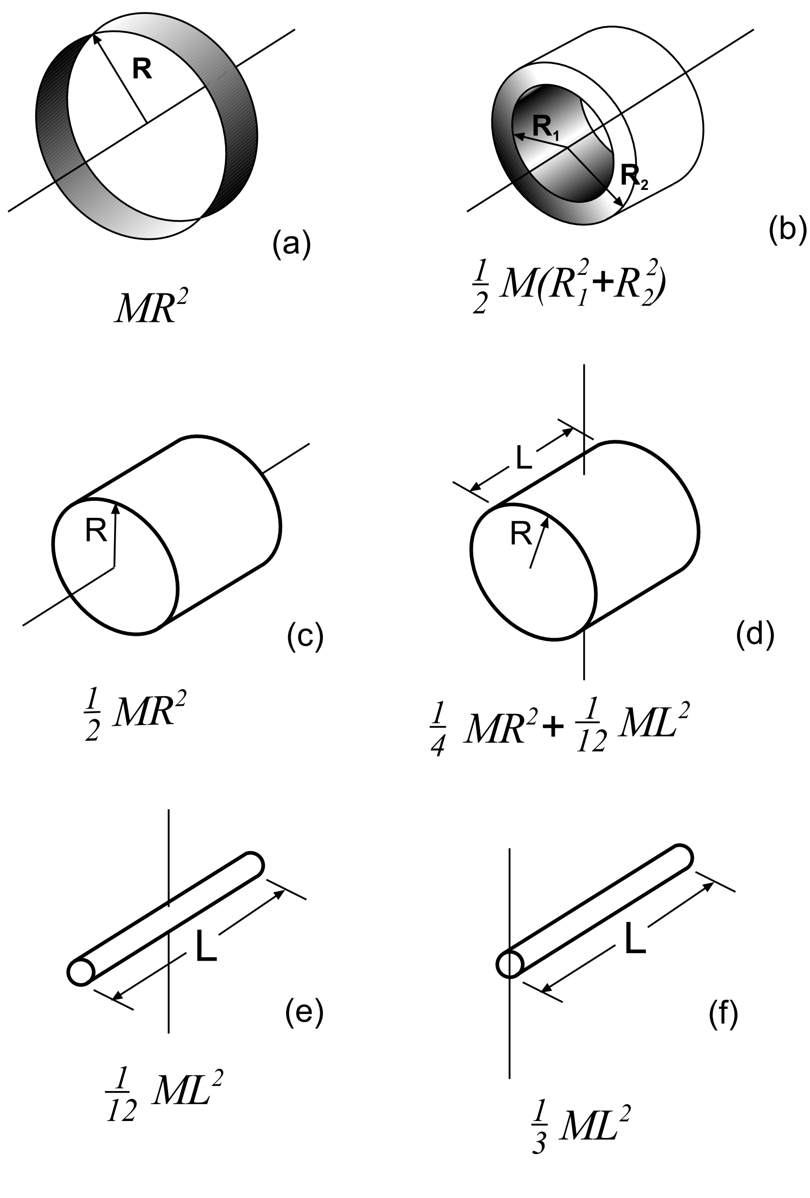

Figure 9.2: Some values of the moment of inertia for various shapes and choices of axes.

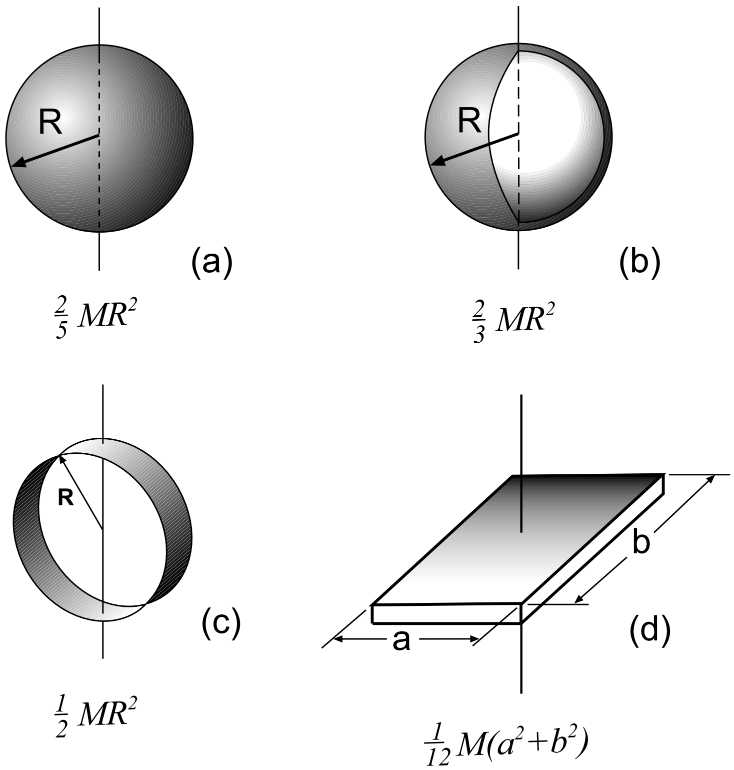

Figure 9.3: More moments of inertia.

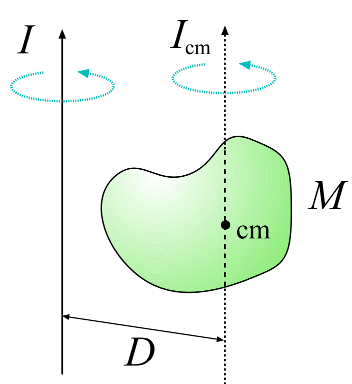

Figure 9.4: The Parallel Axis Theorem: Axis through the cm gives $I_{\rm cm}$, and we want the moment of inertia I about a new axis.

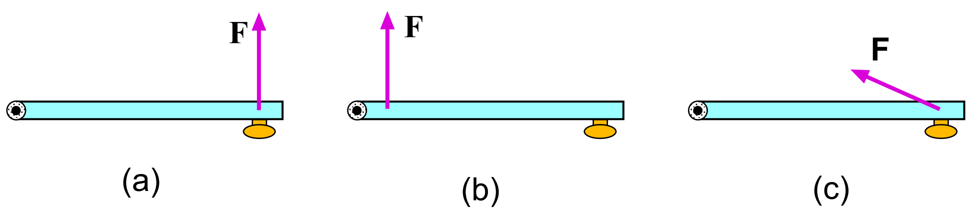

Figure 9.5: Forces exerted on a door: (a) Force exerted far from the hinge, perpendicular to the door face. (b) Force is exerted perpendicular to the door face but close to the hinge! (c) Force is exerted far from the hinge but not perpendicular to the door face. Attempting to open a door using the forces in (b) or (c) will only make you look foolish.

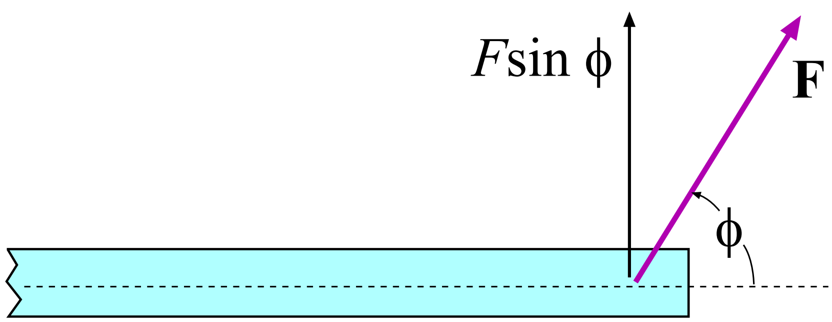

Figure 9.6: Only the perpendicular part of the force, $F_\perp$ contributes to the torque; $F_\perp = F\sin\phi$.

Fig. 9.5(a) shows how one should exert a force to open a door. Push on the end far away from the hinge (axis) and push perpendicularly to the line which joins the axis to the point of application. Exerting a force this way gives a large amount of *torque* on the door.

Figs. 9.5(b) and (c) show how *not* to open a door. In (b) the force is exerted perpendicular to the length of the door but it is too close to the hinge. In (c) the force is exerted far from the hinge but *not* perpendicularly to the line joining hinge and the point of application.

When *you* go opening doors, exert a force like the one shown in 9.5(a)!

The "thing" that makes the door rotate contains the magnitude of the force exerted (F) and the distance from the axis (r), but is only the *perpendicular* part of the force ($F_\perp$) that matters, i.e. perpendicular to the line joining the axis and point of application, as shown in Fig. 9.6.

The magnitude of the torque, τ is the product of $F_\perp$ and r:

$$\tau = rF_\perp = rF\sin\phi \tag{9.6}$$

where ϕ is the angle between the line from the axis the direction of the force.

We now work on the details of the definition of "torque"; it's a little confusing.

Is it a vector or a scalar? In actuality it's a vector, and if the force and line from the axis lie in a certain plane, the direction of the torque is *perpendicular to that plane*, (which

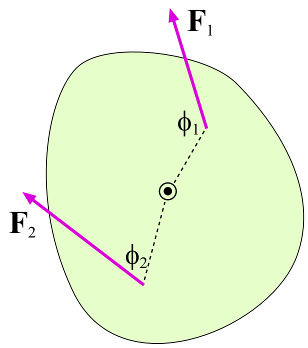

Figure 9.7: Given that our positive rotation direction is counter-clockwise, force $\mathbf{F}_1$ gives a positive torque. Force $\mathbf{F}_2$ gives a negative torque.

we might call the z direction). For our present purposes, the rotations we consider are all simple so we are *only* using the z component of the torque. In that case we're only talking about a single number so we will treat torque as a number. But this number can be positive or negative.

What about the units? From its definition the units must be those of force times those of distance, i.e. $\mathrm{N} \cdot \mathrm{m}$. Now it is true that in Chapter 6 we said that this combination was a "joule". It wouldn't be quite right to use that notation here because torque is a very different quantity from energy and the two quantities never really mix in any of our work. So it is considered good practice to leave the units of torque as $\mathrm{N} \cdot \mathrm{m}$.

Now we need to be more careful about the definition in Eq. 9.6. It is really the *magnitude* of the torque exerted on the object by the force F. Just as our angular displacements can be positive or negative, torque will also bo positive or negative depending on whether the force is making the object rotate in the positive or negative sense.

To avoid using more mathematics than we need, we'll use the following convention: We can use 9.6 to get the magnitude of the torque; if the force would make the object rotate in the counter-clockwise direction, the torque is *positive*. If the force would make the object rotate in the clockwise sense, the torque will be *negative*. Examples are shown in Fig. 9.7.

Of course, we can let "clock-wise" be the positive rotation direction, as long as we are consistent.

With this in mind, we define the magnitude of torque as

$$|\tau| = rF \sin\phi \tag{9.7}$$

where ϕ is angle between the direction of the force and the line from the axis.

When we have several forces acting on a rotating object we will want to find the *total torque*. Just add up the individual torques, making sure you get the signs right.

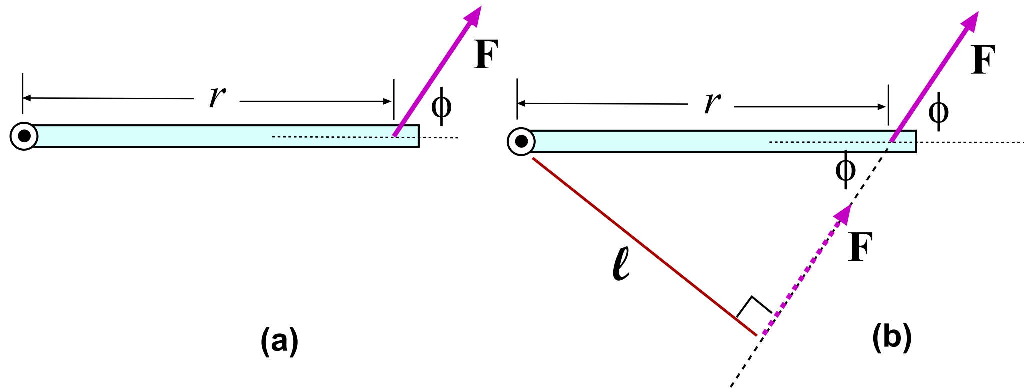

Figure 9.8: (a) Force **F** pulls at angle ϕ from the radial line. (b) We drop a “line of action” along the direction of the force and make a perpendicular “lever arm” from the axis. Now the (same) force pulls perpendicularly to the lever arm. Lever arm has length $\ell = r\sin\phi$.

9.1.5 Another Way to Look at Torque

Another way to arrive at our definition of torque, Eq. 9.7, advocated in some textbooks is as follows.

Again start with a rotating system like a door with a force of magnitude F pulling at a distance r and angle ϕ, as in Fig. 9.8(a). We know that things are simple when the force is pulling perpendicularly to the line from the axis, and we can do something interesting to create this condition.

Fig. 9.8(b) show how to do this. Draw a line along the direction of the force such we can drop a new line from the axis which *is* perpendicular to the force. Now we have the simple situation for torque. The length of this new line is ℓ and is called the **lever arm** for the force in question. We then want to have $\tau = \ell F$ for our formula. But is that the same as what we had before?

Fig. 9.8(b) shows that it is. Since $\ell = r\sin\phi$ we get

$$\tau = \ell F = (r\sin\phi)F = rF\sin\phi$$

and it’s the same as before.

9.1.6 Newton’s 2nd Law for Rotations

What is torque good for? If the torque on a object is greater then the angular acceleration that the object will undergo is greater; that much is obvious, but what is the relation between the two?

We can get a *hint* from considering the motion of single mass point m around an axis at a distance r. Suppose a single force F acts on this mass in the tangential direction.

Then Newton's 2nd law gives $F = ma$, a being the acceleration in the tangential direction. Multiply both sides by r and get $rF = mar$.

Now make some substitutions in this last equation. Since the force is applied at 90° to the radial line, it gives a torque $\tau = rF$. Also the linear acceleration a is *tnagential* and is related to its angular acceleration by $a = a_T = r\alpha$. This gives us:

$$\tau = m(r\alpha)r \qquad \Longrightarrow \qquad \tau(mr^2)\alpha$$

Now the moment of inertia for a *single point mass* m at a distance r is $I = mr^2$. Making this substitution in the last equation gives

$$\tau = I\alpha$$

While this little derivation doesn't show very much it turns out that one can show that the result is general: When we have a bunch of (external) forces acting on a rotating object giving some net torque $\tau_{\rm net}$, the net torque is related to the moment of inertia and angular acceleration by

$$\tau_{\rm net} = I\alpha \tag{9.8}$$

This relation is often called **Newton's 2nd law for rotations** and indeed it strongly resembles the original version of Newton's 2nd law: Compare $F = ma$ with $\tau = I\alpha$. In the second one, torque plays the role of force and the moment of inertia plays the role of the mass. Of course, α corresponds to a as we saw in the last chapter.

9.1.7 Solving Problems with Forces, Torques and Rotating Objects

Knowing how rotating objects behave we can now solve problems which involve (idealized) rotating systems. Often the problem will include a pulley or wheel over which a string passes. The pulley will have *mass*, and that will be an important aspect of the problem but generally we will ignore any friction in the bearings of the pulley. (When we consider it, it will give a torque which opposes its rotational motion.)

We will now have to make free–body diagram (i.e. draw the damn picture) for the rotating objects as well. For the rotating elements the diagram will show the forces acting and *where they are applied*.

Two typical elements in these problems are shown in Fig. 9.9. In (a) a string is wrapped around a pulley or radius R. There is a tension T in the string; the string rolls off the pulley tangentially. We can treat this system as if a force of magnitude T is pulling at the edge of pulley, at right angles to the radial line. Then the torque on the pulley (from the string force) is $TR \sin 90^\circ = TR$.

We will use the fact that the linear motion of the edge of the pulley (where the string rolls off) is the same as that of the string itself.

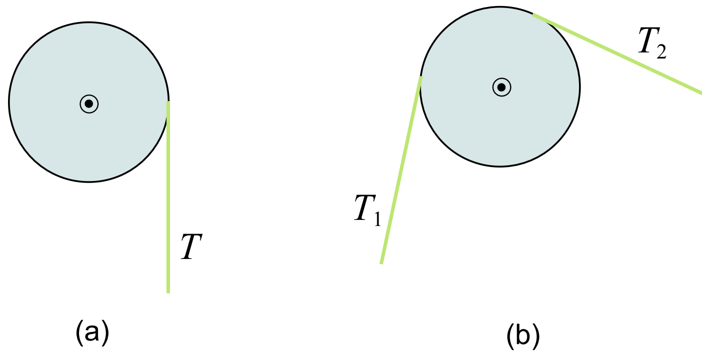

Figure 9.9: (a) String wrapped around a pulley; tension T gives a torque on the pulley. (b) String is in contact with pulley; tensions on the different sides are *not* the same.

Another one is shown in (b). Here a string passes over an ideal pulley and *does not slip*, so again the linear motion of the edge of the pulley is the same as that of the string. Now, in the case that the pulley has mass, the parts of the string on either side of the pulley have *different tensions*. the string forces can be treated as forces applied at the edge, tangent to the pulley, so that in (b) the (clock-wise) torque from the string is

$$\tau = T_1 R - T_2 R = (T_1 - T_2)R$$

9.1.8 An Example

The following example will give show how we can solve problems involving rotating objects.

A string is wrapped around a pulley of mass M and radius R. The string is attached to a mass m; the mass is released. Find the acceleration of the mass as it falls. Treat the pulley as if it were a uniform disk of radius R.

The basic situation is shown in Fig. 9.10. The mass will fall with an acceleration that we expect will be somewhat less than g. As it falls the pulley will turn and since the motion of the edge of the pulley is the same as that of the string, the edge of the pulley will have a tangential acceleration and the pulley will have an angular acceleration; it will rotate faster and faster.

We have to analyze the force with diagrams; we will now have a diagram for the block and one for the pulley. First, the block. Forces on the block are shown in Fig. 9.11(a). Gravity mg points down and the string tension T pulls upward. Since the motion of the mass is downward, we'll let "down" be the positive direction for simplicity and let a be the *downward* acceleration of the mass. Then Newton's 2nd law gives:

$$mg - T = ma \tag{9.9}$$

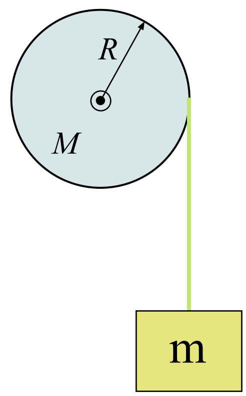

Figure 9.10: Mass hangs from a string which is wrapped around a wheel.

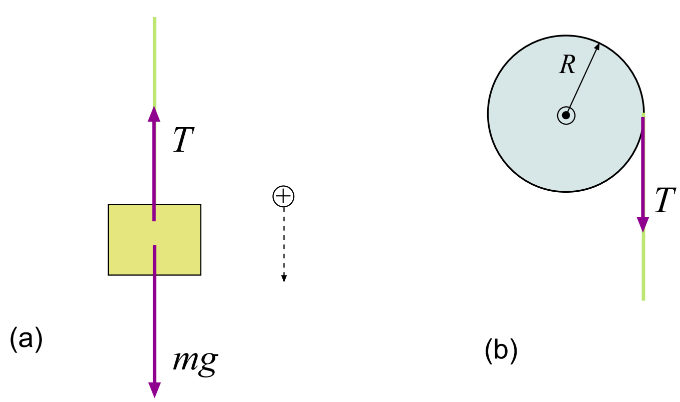

Figure 9.11: (a) Forces on the block. Positive direction of motion will be taken as *downward* for simplicity. (b) Force on the wheel.

Now look at the forces acting on the pulley, shown in Fig. 9.11(b). The string tension acts to give a tangential force T at the edge of the wheel, and hence a torque of RT. That's the only force *which gives a torque around the wheel's axis*. Note, for this picture, we are taking "clock-wise" as the positive rotation direction. (I know, that isn't our convention. So sue me.) This choice makes things consistent with the choice (downward) for the block's motion. Anyways, Newton's 2nd law for rotation gives

$$\tau = RT = I\alpha \tag{9.10}$$

We can treat the pulley as a uniform disk of radius R, so we can substitute $I = \frac{1}{2}MR^2$ here. Also, we recall that the tangential acceleration of the edge of the wheel is the same as that of the string and of the mass m, namely a. Then we can write:

$$a \;=\; a_T \;=\; R\alpha \qquad \Longrightarrow \qquad \alpha \;=\; \frac{a}{R}$$

Now make these two substitutions in Eq. 9.10. Then we have:

$$RT = \left(\tfrac{1}{2}MR^2\right)\frac{a}{R} = \tfrac{1}{2}MRa \qquad \Longrightarrow \qquad T = \tfrac{1}{2}Ma \tag{9.11}$$

Use this to substitute for T in Eq. 9.9 and get

$$mg - \tfrac{1}{2}Ma = ma \tag{9.12}$$

Now do some algebra to solve for a:

$$mg \;=\; \left(\tfrac{1}{2}M + m\right)a \qquad \Longrightarrow \qquad a \;=\; \frac{mg}{\left(\tfrac{1}{2}M + m\right)} \tag{9.13}$$

This result makes sense if we set $M = 0$. We find that the formula then gives $a = g$. This is what we *should* get because that is the case where the hanging mass is really attached to nothing and falls freely under gravity.

9.1.9 Statics

From Chapter 7 and the current chapter we have some rules about the total forces and torques on objects which, for our present purposes can be expressed as:

If the center of mass of an object is at rest, then the total external force on the object must be zero.

If an object is not rotating then the total torque on it must be zero.

Actually the second of these statements is true for *any* choice of an axis which as we'll see can be used to make problem–solving a little easier.

Now, there many situations in the world when we we'd like some object to be absolutely motionless. For example, since gravity acts on all objects on the earth we might need to hold them up and make sure they don't fall; we need to do this with various forces of support and we need to know what forces are required to support the objects.

Problems of this sort can be very complicated if the structures and the applied forces are messy; there is a whole area of engineering devoted to this, known as "Statics". We will work with some very simple examples of this sort of problem.

The strategy for solving such problems is to first draw the force diagram for the object. (You *can't* do these problems without a diagram.) Include the dimensions of the objects and where the forces are applied, and if possible the directions of the forces. Then apply the conditions:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau = 0 \tag{9.14}$$

(We'll only work in two dimensions so the force condition just has two components.) As mentioned, for the condition on the torques you can choose any point on the object to server as an axis.

To make the math easier it is often useful to put the axis at a place where one or more unknown forces are applied; since those forces will give no *torque* about that axis they will not appear in the torque equation.

9.1.10 Rolling Motion

A very common kind of motion is when a round object like a cylinder or a sphere **rolls without slipping** on a surface. This means that as the object moves the arc length around its edge matches one-to-one with the distance travelled linearly, as indicated in Fig. —.

Because of this match-up, we can relate the quantities for linear motion with the angular quantities. First, if the object turns through an angle θ while rolling the linear distance travelled by the center of the object is

$$x_c = R\theta \tag{9.15}$$

where R is the radius of the object. The velocity of the center and angular velocity of the object are related by

$$v_c = R\omega \tag{9.16}$$

And finally the linear acceleration of the center and angular acceleration of the object are related by

$$a_c = R\alpha \tag{9.17}$$

These are the values of the linear speed, velocity and acceleration of the *center* of the object. The instantaneous velocities of the other parts of the object are different: The point in contact with the surface has an instantaneous velocity of *zero* which the top point has a speed of $2v_c$ in the forward direction.

Because the rolling object is in contact with the surface there may be a force of friction from the surface. If so, it is a force of *static* friction because the very bottom of the rolling object has no velocity relative to the surface. (And recall that the normal force of the surface only gives us the *maximum* value of the static friction force.)

A rolling object has kinetic energy, of course. One can show that if an object with moment of mass M and moment of inertia I is rolling so that the speed of its center is v_c and its angular velocity is ω, its kinetic energy is

$$\mathrm{KE} = \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}I\omega^2 = \mathrm{KE}_{\mathrm{trans}} + \mathrm{KE}_{\mathrm{rot}} \qquad (9.18)$$

that is, it is the sum of two parts: One part comes from the translational motion of the center, and the other comes from the rotational motion of the object about its center.

Sometimes in solving a problem, the total KE in Eq.9.18 for a rolling object can be expressed more simply if we know the expression for the moment of inertia. For example, suppose the object is a uniform cylinder. In that case, $I = \frac{1}{2}MR^2$ and since $\omega = v_c/R$ we get

$$\begin{aligned}
\mathrm{KE}_{\mathrm{total}} &= \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}I\omega^2 \\
&= \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}(\tfrac{1}{2}MR^2)\left(\frac{v_c}{R}\right)^2 \\
&= \tfrac{1}{2}Mv_c^2 + \tfrac{1}{4}Mv_c^2 \\
&= \tfrac{3}{4}Mv_c^2
\end{aligned}$$

9.1.11 Example: Round Object Rolls Down Slope Without Slipping

An important example of rolling motion is that of a round symmetrical object rolling down a slope inclined at angle $|theta$, as illustrated in Fig. 9.12 (if for no other reason than that it is easy to set up in the lab). The object has mass M, radius R and moment of inertia I about its center.

We would like to find the acceleration of the center of the object. Recall that when we did this problem for a mass *sliding* down the slope without friction we got $a = g\sin\theta$; but this is a different problem.

The forces acting on the object are shown in Fig. 9.13. Gravity acts with magnitude Mg downward and we can treat its force as acting at the center of the object. (Thinking ahead, we split the vector into parts "down the slope" and perpendicular to the slope as shown in Fig. 9.13. Then we work with these components in Newton's laws.) The normal force of

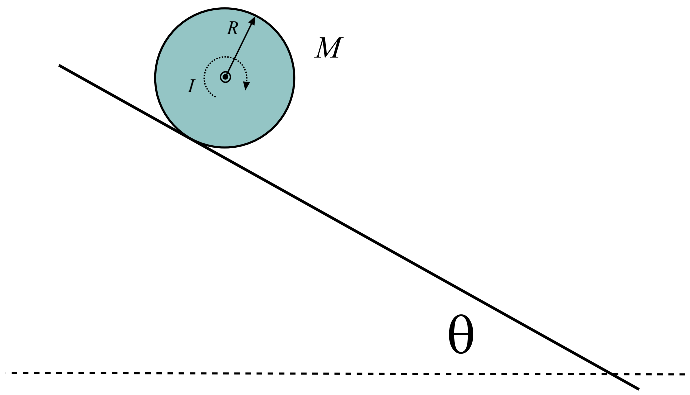

Figure 9.12: Example: Object with mass M radius R and moment of inertia I rolls (without slipping) down a slope inclined at θ above the horizontal.

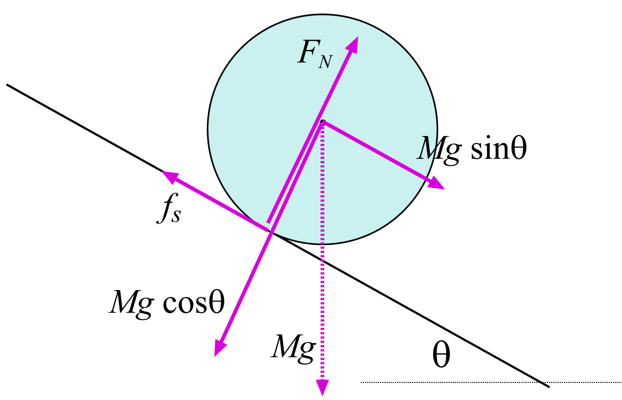

Figure 9.13: Forces acting on the round object as it rolls down the slope. The downward force of gravity Mg has been split up into its components along/perp to the slope.

the surface F_N is applied at the point of contact, as is the force of friction f_s which acts along the surface in the direction shown. (Actually, it may not be clear that friction acts in the backward direction, but that will be clear later.)

We will consider the forces acting on the object and apply the regular version of Newton's 2nd law. Then we will consider the torques acting about the center of the wheel and apply the rotational version of Newton's 2nd law. Eventually we will get a_c.

Take "down the slope" as the positive direction. The forces acting down the slope are the component $Mg\sin\theta$ from gravity *minus* the force of friction f_s which acts up the slope. Newton's 2nd law gives:

$$Mg\sin\theta - f_s = Ma_c \tag{9.19}$$

Next the torque: This time we will take "clock-wise" to be the positive rotation direction (to be consistent with the linear acceleration) and sum of the clock-wise torques will equal $I\alpha$. Since gravity acts *at* the center and the normal force acts along a line through the center (so $\phi = 0$), only the friction force gives a torque about the center. It act perpendicularly to the radial line, at a distance R, so:

$$\tau_{\rm net} = f_s R = I\alpha \tag{9.20}$$

Lastly, by Eq. 9.17, a_c and α are related:

$$a_c = R\alpha \tag{9.21}$$

Equations 9.19, 9.20, and 9.21, have three unknowns (f_s, α and a_c) so we can solve for them. Putting 9.21 into 9.20 gives

$$f_s = \frac{I\alpha}{R} = \frac{I(a_c/R)}{R} = \frac{Ia_c}{R^2}$$

and putting this into 9.19 gives

$$Mg\sin\theta - \left(\frac{I}{R^2}\right)a_c = Ma_c$$

Doing some algebra to get a_c, we get:

$$Mg\sin\theta = \left(M + \frac{I}{R^2}\right)a_c$$

and finally (drum roll), the answer:

$$a_c = \frac{Mg\sin\theta}{\left(M + \frac{I}{R^2}\right)} \tag{9.22}$$

We can make it a little cleaner by dividing top and bottom by M, and then we have:

$$a_c = \frac{g\sin\theta}{\left(1 + \frac{I}{MR^2}\right)} \tag{9.23}$$

and then we see that since the denominator is bigger than 1 the acceleration of the object must be smaller that $g\sin\theta$, the "sliding" value. Also if we could fix the value of M but make the moment of inertia larger, a_c would get smaller.

Some specific examples might be useful here. If the rolling object is a uniform cylinder then $I/(MR^2) = \frac{1}{2}$ and we get:

$$a_c = \frac{g\sin\theta}{\left(1 + \frac{1}{2}\right)} = \frac{g\sin\theta}{\frac{3}{2}} = \tfrac{2}{3}g\sin\theta$$

or if the object is a sphere then $I/(MR^2) = \frac{2}{5}$ and then

$$a_c = \frac{g\sin\theta}{\left(1 + \frac{2}{5}\right)} = \frac{g\sin\theta}{\frac{7}{5}} = \tfrac{5}{7}g\sin\theta$$

Note that in both of these results the actual mass and radius do not appear although the results depend on the shapes of the objects; *any* uniform cylinder will roll down the slope with acceleration $\frac{2}{3}g\sin\theta$. Since $\frac{5}{7} > \frac{2}{3}$ a uniform sphere has a larger acceleration than the cylinder in rolling down the slope.

9.1.12 Angular Momentum

And we conclude with the rotational version of momentum.

We have found that the mass m from linear dynamics corresponds to the moment of inertia I in rotational dynamics, and the velocity v corresponds to the angular velocity ω. We found in Chapter 7 that a useful quantity in linear motion was the momentum, $p_x = mv_x$. If we had to come up with the rotational version of this quantity (for whatever reason) it would have to be $I\omega$, and that is what we will use for the rotational version of "momentum" . For our simple rotating systems, we define the **angular momentum** L as:

$$L = I\omega \tag{9.24}$$

Angular momentum (as we will use it, for our simple rotations) is just a single number, although as with angular velocity and torque we're really talking about the z component of a vector. Its units are those of I times those of ω (without the "rad" marker, i.e. $\frac{1}{\mathrm{s}}$), thus they are:

$$\mathrm{kg}\cdot\mathrm{m}^2\cdot\frac{1}{\mathrm{s}} = \tfrac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}}$$

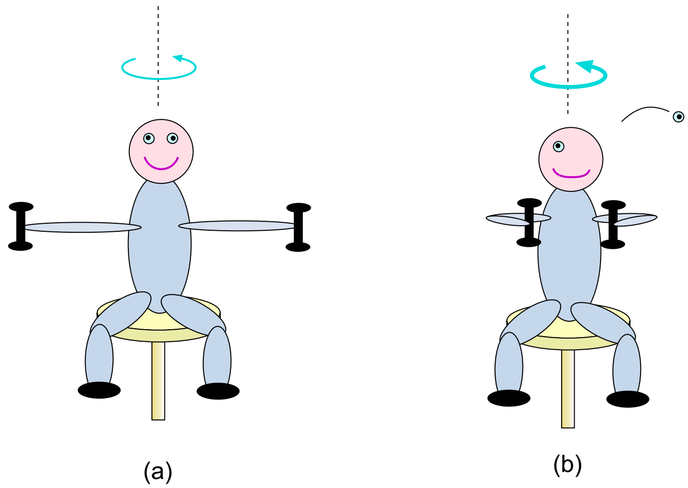

Figure 9.14: (a) Minimalist man holds weights in outstretched arms. Initially, he has a small rate of rotation. (b) Man pulls weights inward; moment of inertia of system is smaller and rate of rotation is larger.

Again, there is no common abbreviation for this unit.

We will refer to our previous quantity $\mathbf{p} = m\mathbf{v}$ as *linear* momentum to keep things straight.

Recall that the *reason* that linear momentum was so useful was that it was *conserved* for an isolated system. That is, if the net external force acting on a system of particles was zero, their total linear momentum stayed the same.

A similar thing is true for the rotational case; the terms just translate over from linear to rotational language:

If there is no net external torque on a system of objects, their total angular momentum stays the same.

This principle is known as the **conservation of angular momentum**.

A familiar example of the application of this principle is that of a spinning ice skater who is initially turning at a slow rate with her arms outstretched and then turns much faster when she pulls them in. One can also show the effect by resting on a turntable or rotating stool and holding two weights in your outstretched arms, as shown in Fig. 9.14. If you are given a small rotation rate at first you can make yourself spin much faster by pulling in your arms, so much so that it is often hard to stay on the turntable!

What is going on here?

Both before and after the arms are pulled in, the angular momentum of the turning system is given by $L = I\omega$. Since there are no (significant) external torques on the system, the product of L and ω will *remain the same*. But when the arms are pulled in the moment of inertia I decreases so then ω — the rate of turning — must increase.

Using some math, if I_i and I_f are the initial and final moments of inertia and ω_i and ω_f are the initial and final angular velocities, then conservation of angular momentum gives

$$I_i\omega_i = I_f\omega_f$$

so that the final angular velocity is

$$\omega_f = \left(\frac{I_i}{I_f}\right)\omega_i$$

that is, the final angular velocity is bigger than the initial angular velocity by a factor given by the ratio of the moments of inertia.

A word of caution about this example: It is true that *angular momentum* is conserved as I changes, but in general the kinetic energy is *not* conserved. The rotating man has more kinetic energy after pulling his arms inward. You can appreciate this if you do the demonstration yourself; you will feel yourself *doing work* as you pull your arms inward. increasing the energy of the rotating system.

9.2 Worked Examples

9.2.1 The Moment of Inertia and Rotational Kinetic Energy

1. A horizontal 150–kg merry-go-round of radius $2.0\,\mathrm{m}$ **is turning at a rate of** $32.0\,\mathrm{rpm}$**. What is its kinetic energy? (Assume the merry-go-round is a uniform disk.)**

Find the angular velocity of the merry-go-round: Converting from revolutions per minute to radians per second,

$$(32.0\,\tfrac{\mathrm{rev}}{\mathrm{min}})\left(\frac{1\,\mathrm{min}}{60\,\mathrm{sec}}\right)\left(\frac{2\pi\,\mathrm{rad}}{1\,\mathrm{rev}}\right) = 3.35\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

Using the formula for the moment of inertia of a uniform disk from Eq. 9.4, we have:

$$I = \tfrac{1}{2}MR^2 = \tfrac{1}{2}(150\,\mathrm{kg})(2.0\,\mathrm{m})^2 = 300\,\mathrm{kg}\cdot\mathrm{m}^2$$

and from Eq. 9.3, the kinetic energy is

$$\mathrm{KE} = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(300\,\mathrm{kg}\cdot\mathrm{m}^2)(3.35\,\tfrac{\mathrm{rad}}{\mathrm{s}})^2 = 1.68\times10^3\,\mathrm{J}$$

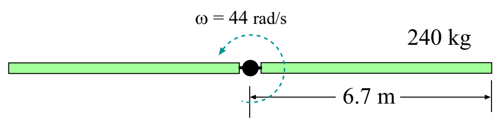

Figure 9.15: Helicopter blades for Example 2.

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2. A helicopter has two blades each of which has a mass of $240\,\mathrm{kg}$ and can be approximated as a thin rod of length $6.7\,\mathrm{m}$. The blades are rotating at an angular speed of $44\,\frac{\mathrm{rad}}{\mathrm{s}}$. (a) What is the total moment of inertia of the two blades about the axis of rotation? (b) Determine the rotational kinetic energy of the spinning blades. [CJ7 9-46]

(a) We sketch the system in Fig. 9.15. From our formulae for the moment of inertia of basic shapes, we recall that the moment of inertia of a thin stick rotating about an axis *at one end of the stick* is $\frac{1}{3}ML^2$ where L is the length of the stick. We have *two* such sticks in our simple treatment of the helicopter blades, so the total moment of inertia is

$$I_{\mathrm{blades}} = 2\left(\tfrac{1}{3}ML^2\right) = 2\left(\tfrac{1}{3}(240\,\mathrm{kg})(6.7\,\mathrm{m})^2\right) = 7.18 \times 10^3\,\mathrm{kg}\cdot\mathrm{m}^2$$

(b) And now to get the kinetic energy of the rotating blades, use Eq. 9.3:

$$\mathrm{KE} = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(7.18 \times 10^3\,\mathrm{kg}\cdot\mathrm{m}^2)(44\,\mathrm{s}^{-1})^2 = 6.95 \times 10^{-6}\,\mathrm{J}$$

Chapter 10

Oscillatory Motion

10.1 The Important Stuff

10.1.1 Introduction

All right, back to one-dimensional motion. (Eh? Haven't we done everything that can be done with one-dimensional motion?)

In Chapter 6 we encountered the "spring force", a force which is proportional to the displacement and *opposes* it. As you will surely recall, the force from an "ideal" spring followed the formula given in Eq. 6.8,

$$F = -kx \tag{10.1}$$

where k was called the "force constant" of the spring and x is the displacement of the end of the spring from the equilibrium position. You'll also recall that the energy stored in the spring is $\mathrm{PE} = \frac{1}{2}kx^2$.

In Chapter 6 we had masses bump into springs and exchange energy with them. Now we'll do something a little different. We'll *attach* a mass to the end of a spring, pull the mass back a little ways (a distance A) and then release it, as shown in Fig. 10.1. (The surface on which the mass is sliding is frictionless!)

What do we expect the ensuing motion to be like?

Since the mass gets pulled inward when the spring is stretched and gets pushed outward when the spring is compressed we expect the motion to be represented by something like the x vs. t graph in Fig. 10.2.

10.1.2 Harmonic Motion

The graph in Fig. 10.2 shows how the mass would move if the spring is ideal and obeys the force law $F_{\mathrm{spr}} = -kx$. The curve is "sinusoidal" meaning that it follows the basic form of

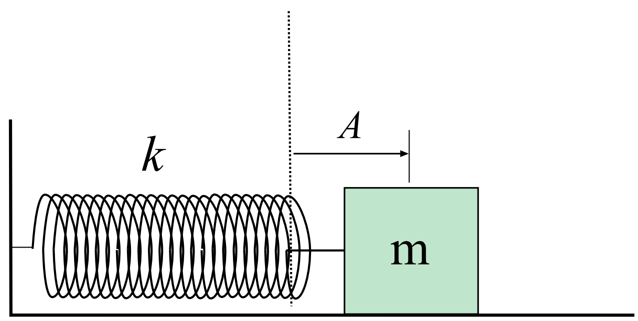

Figure 10.1: Mass m attached to spring of force constant k is pulled back a distance A and released.

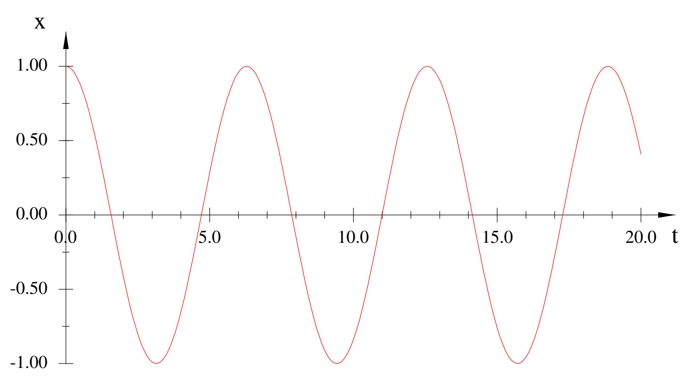

Figure 10.2: Motion of mass on the end of a spring.

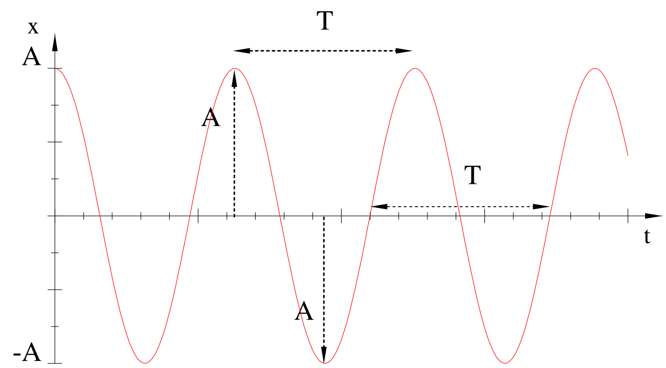

Figure 10.3: Period T and amplitude A of the motion of the mass on a spring.

the sine or cosine functions (I hope) you've seen in math.

When the motion of an object depends on time in this sinusoidal fashion, we say that the object is undergoing **harmonic motion** and that the physical system producing such a motion is a **harmonic oscillator**.

The maximum distance which the mass goes *outward* from the equilibrium point is the same as the maximum distance it goes *inward*. This distance is called the **amplitude** of the motion. In Fig. 10.3 the amplitude is indicated by A. (For the mass's motion the amplitude is a *length* and is measured in meters.)

Then there is the amount of time required for the mass to make one full oscillation, that is, to go back and forth and return to a place where its motion will repeat itself. This is the **period** of the motion, and it is indicated (twice) on Fig. 10.3. The period T is a *time* and is measured in seconds.

A related number describing the rapidity of the oscillations is the number of oscillations the mass will make in a given time period; this is called the **frequency** of the oscillations:

$$f = \frac{(\#\text{ of oscillations})}{\text{time}} \tag{10.2}$$

The frequency is the inverse of *time* divided by *oscillations*; but the time for each oscillation is the period T, so that f is the inverse of T:

$$f = \frac{1}{T}$$

One must be careful with the units of f. Even though it is the inverse of a time one should *always* express f in terms of oscillations (or cycles) per second. It is true that "oscillations" is not really a fundamental unit but it is important to distinguish *frequency* from *angular velocity* ω because we will find that the two quantities are related.

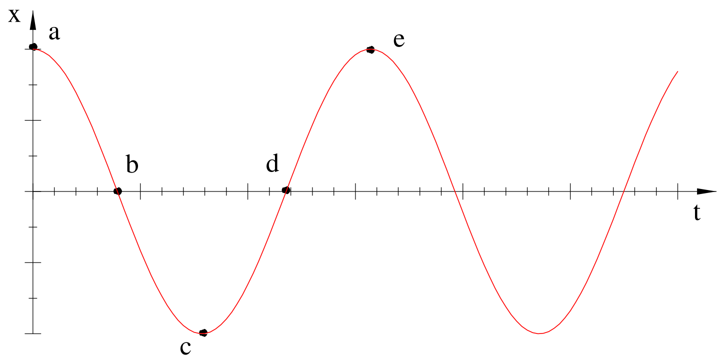

Figure 10.4: Points

In place of $\frac{\text{cycles}}{\text{s}}$ we can also use shorter notation "Hertz"; thus:

$$1\,\text{Hz} = 1\frac{\text{cycle}}{\text{s}} = 1\frac{\text{osc}}{\text{s}}$$

10.1.3 Displacement, Velocity and Acceleration

We now look at the graph of x vs. t in greater to understand what is happening with the position, velocity and acceleration of the mass.

In Fig. 10.4 the plot of the motion is given over a couple cycles with some important points marked (a)–(e).

At point (a) the mass is "released", meaning that its initial velocity is zero. At this point the displacement is a maximum. The acceleration of the mass is negative and takes on its maximum size (magnitude) at this point because the spring force has maximum size at this point.

At point (b) the displacement is zero; here the mass is zipping through the equilibrium position, where the spring has its natural length and exerts no force. the velocity is negative and has its maximum size at this point. The acceleration is zero here.

At point (c) the mass has gone as far inward as it's going to go; the displacement here is negative and has the maximum size. The velocity is zero at this point. The acceleration is positive and has its maximum value.

At point (d) the mass is once again at the equilibrium point so that the displacement x is zero. But the velocity is now *postivie* and has the same maximum size as it did before at this point. Since the spring is undistorted here the force and acceleration are zero.

At point (e) the motion is the same as it was at (a), i.e. maximum displacement, zero velocity, maximal (negative) acceleration.

From here on the motion repeats.

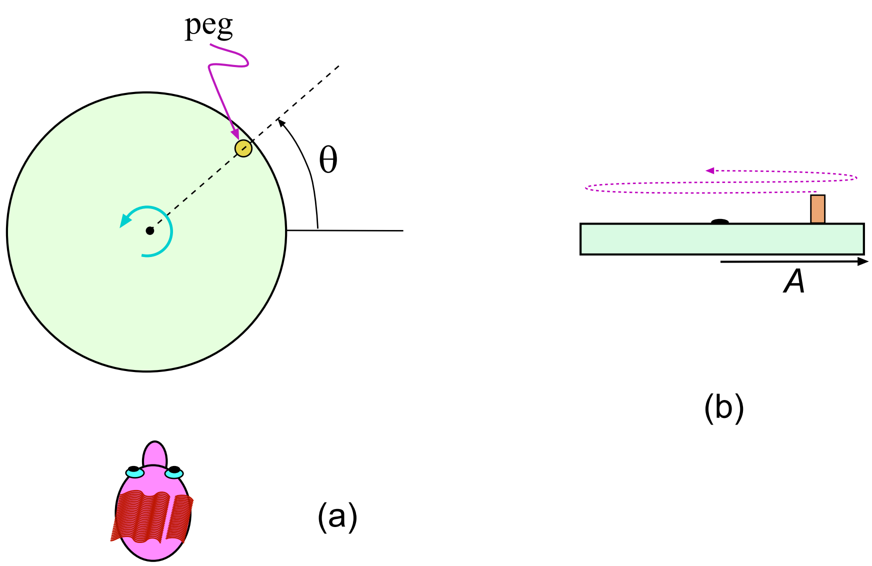

Figure 10.5: Reference circle. (a) Top view of you with your bad haircut looking at the turning wheel along its edge. Wheel has a peg located at its edge, at $r = A$. (b) What you see when you look at the wheel; peg seems to move back and forth along a line.

10.1.4 The Reference Circle

We can get some useful insight into the motion of the oscillating mass by relating it to the motion of a mass in uniform circular motion.

Imagine that we have a wheel of radius A which turns at a constant rate; at the edge of the wheel is a peg. Then imagine that you are looking at the turning wheel *from the side*, i.e. level with the surface of the wheel. This is shown in Fig. 10.5(a). What *you* would see in shown in Fig. 10.5(b); you would see the profile of the peg moving back and forth horizontally, oscillating much like the mass on the of the spring. . . except that here there's no spring!

It turns out that the motion of the peg is *exactly* like that of the mass on the spring; in both cases the object moves between $x = -A$ and $x = +A$ with a sinusoidal dependence on time. For the case of the peg we can see this with a little math. Suppose the reference wheel turns at a constant angular velocity ω then (as usual) assuming $\theta = 0$ at $t = 0$, then θ is given by $\theta = \omega t$.

Now the x coordinate of the peg (which is all the man in Fig. 10.5(a) can see) is given by $x = A\cos\theta$ (see Fig. 10.6.) Putting these relations together, we have

$$x = A\cos(\omega t)$$

and for this relation the graph of x vs. t is a sinusoidal curve.

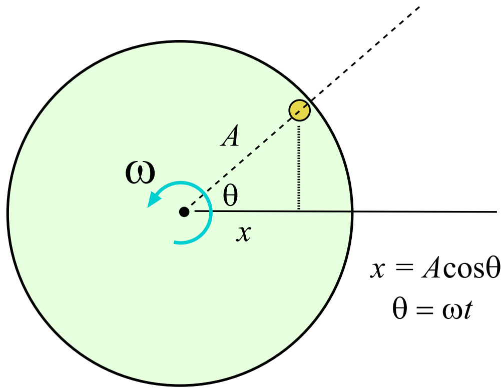

Figure 10.6:

Thinking of the reference circle we can *associate* an angular velocity ω with the motion of an oscillator. This makes sense even though the mass in question really moves back and forth along a line, *not* in a circle.

The angular velocity ω we associate with the oscillator and its frequency are related. Every time the mass on the spring makes one full cycle the corresponding peg on the wheel turns through 2π radians. This means that the angular velocity ω (in radians per second) must be 2π times as large as the frequency f (in cycles per second) so that:

$$\omega = 2\pi f \tag{10.3}$$

But here I will repeat the caution about the units given before: Eq. 10.3 seems to say that both ω and f have the same units, but to avoid confusion we should always express f in $\frac{\text{cycles}}{\text{s}}$ or Hz.

When we speak about an oscillator we say that ω is the **angular frequency** of the oscillator, as distinguished from the plain old frequency f.

We can use the reference circle to derive some useful formulae about oscillatory motion from the fact that the man watching the peg on the wheel sees only the *sideways* (x) part of the peg's motion, and thus only the x components of its velocity and acceleration vectors. When the peg is at the $\theta = 90^\circ$ position, as shown in Fig. 10.7(a), the velocity vector (which always has the same magnitude, namely $v = \omega A$)) is pointing sideways so that the man sees the mass with a speed of $|v_x| = v_{\max} = \omega A$. So the largest speed of the mass is related to the amplitude and frequency by

$$v_{\max} = \omega A = 2\pi f A \tag{10.4}$$

Next, consider the "view" of the mass when it is at the $\theta = 0^\circ$ position, as in Fig. 10.7(b). The man "sees" zero velocity but he sees the *maximum* size of the acceleration. The acceleration vector of the peg always has magnitude $a_c = \frac{v^2}{A}$ and here the man sees a sideways

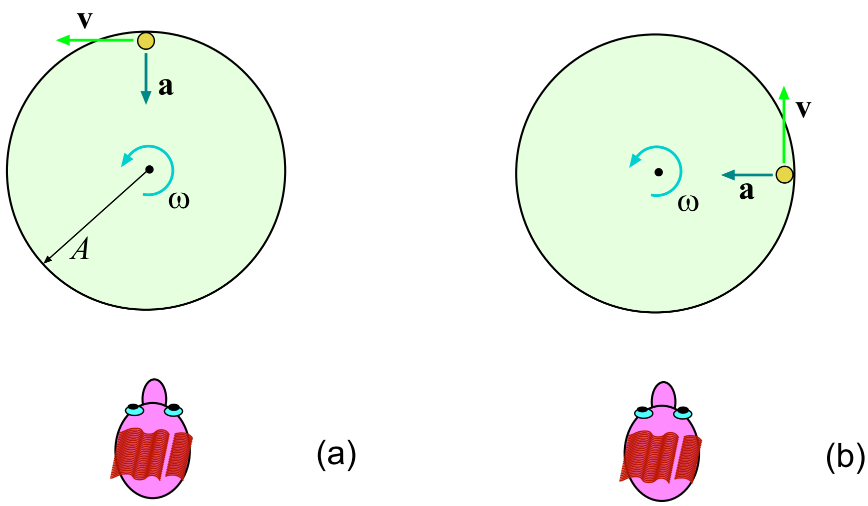

Figure 10.7: Man with big nose and bad haircut can only "see" the *sideways* components of the velocity and acceleration vectors. (a) When the peg is at the $\theta = 90°$ position man sees the peg with maximum velocity and zero acceleration. (b) When peg is at the $\theta = 0°$ position man sees the peg with zero velocity and maximal acceleration.

acceleration a_x of this size. So the maximum acceleration is

$$|a_{x,\,\mathrm{max}}| = \frac{v^2}{A}$$

But now using $v = \omega A$ we get

$$|a_{x,\,\mathrm{max}}| = \frac{v^2}{A} = \frac{(\omega A)^2}{A} = \omega^2 A \tag{10.5}$$

But the sideways motion of the peg is the same as that of the corresponding mass on a spring, and when the mass is at the "far" position (the 0° position for the peg) the force of the spring has magnitude kA. Then from Newton's 2nd law the acceleration at the "far" position the value

$$|a_x| = |F_x|/m = \frac{kA}{m}$$

which is the same as the a_{max} in the equation before it. Equating the two we find

$$\omega^2 A = \frac{kA}{m} \qquad \Longrightarrow \qquad \omega = \sqrt{\frac{k}{m}} \tag{10.6}$$

and so a mass m oscillating on a spring of force constant k the frequency of the motion is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{10.7}$$

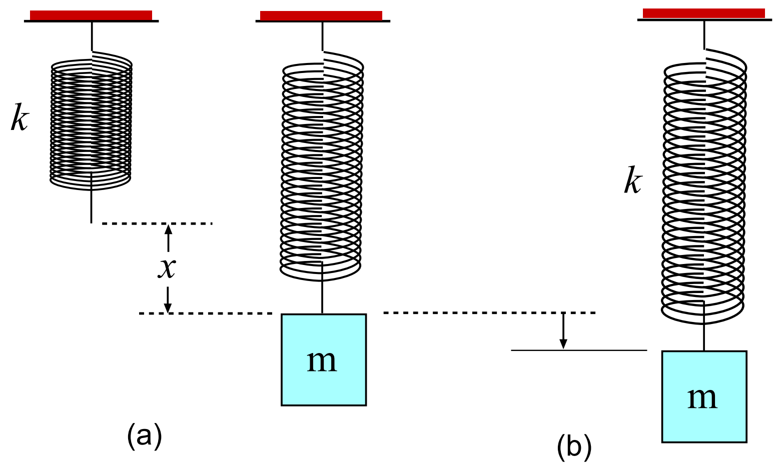

Figure 10.8: (a) Mass is attached to the end of a vertical spring; the spring stretches from its original length so as to support the mass. (b) Mass will now oscillate around the new equilibrium position.

and the period of the motion is

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{10.8}$$

The answer is a little surprising because the amplitude of the motion (A) does not appear in it. So if we start the oscillations by pulling the mass back by *any* distance A the frequency of the oscillations is the same! (Recall the assumption about the spring behaving *ideally*; if the spring doesn't follow $F_x = -kx$ then we can't make this statement.)

It might seem as though with a larger amplitude the period would be greater (the mass has to move farther), but that's not the case!

10.1.5 A Real Mass/Spring System

Since the mass on the spring in Fig. 10.1 needs to slide on a frictionless surface, the setup illustrated would be difficult to set up in the lab even if we could get a spring which has ideal behavior. With friction present the system would lose mechanical energy and come to a halt.

It turns out that one can get (basically) the same behavior by suspending the mass from a spring and letting the mass bob up and down as shown in Fig. 10.8. When the mass hangs from the spring with no motion, as in Fig. 10.8(a), the spring is stretched so as to give a force to hold the mass up, i.e. the spring force equals the weight of the spring. The elongation x is given by

$$|F_{\rm spr}| = F_{\rm grav} \qquad \Longrightarrow \qquad kx = mg \qquad \Longrightarrow \qquad x = \frac{mg}{k}$$

Now if mass is given a little tap it will oscillate up and down around this *new* position (as shown in Fig. 10.8(b)) with the *same* frequency as in the horizontal case, that is, $T = 2\pi\sqrt{m/k}$. It might seem surprising that the frequency doesn't depend on the value of g, but it doesn't. The effect of gravity is that the mass is oscillating about a point other than the *true* equilibrium length of the spring.

So you *can* set up a mass/spring system in the lab just as long as the spring behaves ideally under the conditions of your experiment. If you hang an enormous mass from the spring it may deform and fail to act ideally. Even worse, you will have to pay for the damaged spring if they find out who did it.

One word of warning about using the frequency formula Eq. 10.8 for a real spring (either horizontal or vertical). It assumes that the mass of the spring is very small compared to that of the hanging mass, but that may not be the case. The bits of the spring are also moving up and down —the part closest to the mass has the greatest movement— and as a result the *effective* mass of this oscillator is greater. For reasons far too long to go into, it is correct to include $\frac{1}{3}$ of the spring's mass with that of the hanging mass so the equation for the period is really

$$T_{\text{real spr}} = 2\pi\sqrt{\frac{m + \frac{m_{\text{spr}}}{3}}{k}} \tag{10.9}$$

Unless otherwise stated, we'll assume that the springs in the problems will be massless.

10.1.6 Energy and the Harmonic Oscillator

As the mass oscillates on the spring the energy of the mass–spring system stays the same. Suppose the total energy of the system is E_{Tot}. Then when the mass is at $x = \pm A$ (the extreme positions) the speed is zero so there is no kinetic energy. The energy of the system at these points is all contained the potential energy of the spring, which is general is given by $\frac{1}{2}kx^2$. This tells us:

$$\tfrac{1}{2}kA^2 = E_{\text{Tot}}$$

Similarly when the mass is zipping through the equilibrium (central, $x = 0$) position there is no potential energy in the spring and the energy is all kinetic. Since that speed of the mass is v_{max} here, we have:

$$\tfrac{1}{2}mv_{\text{max}}^2 = E_{\text{Tot}}$$

Equating the two expressions, we get

$$\tfrac{1}{2}kA^2 = \tfrac{1}{2}mv_{\text{max}}^2$$

At all *other* points in the motion of the mass there is both potential energy *and* kinetic energy so if the mass is at position x and has velocity v_x, we have

$$\tfrac{1}{2}kx^2 + \tfrac{1}{2}mv_x^2 = E_{\text{Tot}} \tag{10.10}$$

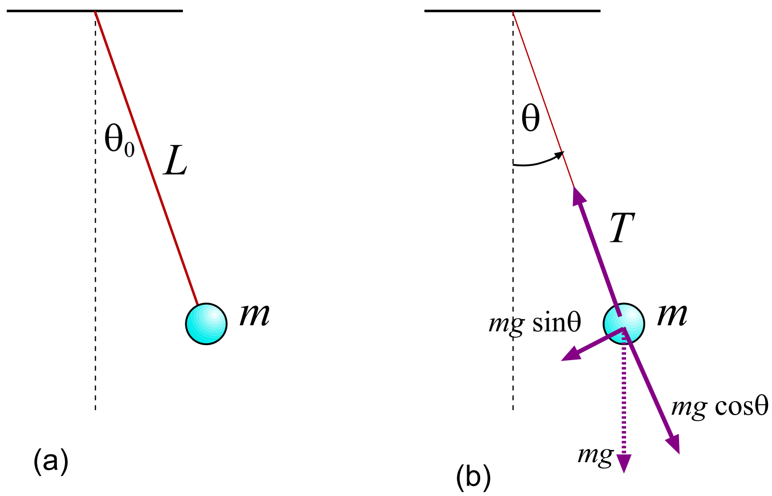

Figure 10.9: (a) Simple pendulum of length L with small mass m attached to the end. Pendulum is pulled back by θ_0 and released. (b) Forces acting on the mass when pendulum is at angle θ from the vertical. The down ward force of gravity mg has been split into components along and perpendicular to the string.

10.1.7 Simple Pendulum

Another oscillating system is shown in Fig. 10.9(a). A small mass m is attached to a light string of length ℓ; it is pulled back by some angle θ_0 and released. The mass swings back and forth in a vertical plane. This is a pendulum, of course; since all the mass in the system is at the very end of the string it is called a **simple pendulum**.

Clearly the mass will swing back and forth between angles of $\pm\theta_0$ from the vertical. (Total energy will remain the same so it will rise to the same height as when it started.) We'd like to find how the period of this motion depends on the physical properties of the pendulum.

We can view the mass and string as a system that rotates about the place where the string is attached. Then since there is only a single point mass a distance L from the axis, the moment of inertia of this system is $I = mL^2$.

The torque on the system is provided by gravity. The forces acting on the mass are shown in Fig. 10.9(b). String tension T pulls inward along the string's length and the downward force of gravity, mg has been split up into components along the string and perpendicular to it. The forces which act along the string give no torque about the axis (the $\sin\phi$ factor for the torque is zero) but the perpendicular part of gravity, $mg\sin\theta$ is exerted at a distance of L and gives a torque of magnitude $\tau = L(mg\sin\theta)$.

We need to get the *sign* of the torque correct here. From Fig. 10.9(b) we see that when θ is positive, the torque τ is in the opposite (negative) sense, so actually we should say:

$$\tau = -mgL\sin\theta$$

From $\tau = I\alpha$ and $I = mL^2$ we then get

$$\tau = -mgL\sin\theta = (mL^2)\alpha$$

and then some algebra gives

$$\alpha = -\left[\frac{g}{L}\right]\sin\theta \tag{10.11}$$

One more step is needed so that we can get f from this equation. It turns out that when the angle θ is small (and measured in radians, as is assumed in Eq. 10.11) the value of $\sin\theta$ is very close to the value of θ itself. If θ is less than the radian equivalent of 20° the two values are within 2% of each other. We will promise (hah!) never to use our results when the angle of swing is greater than 20° and then in place of Eq. 10.11 we will write

$$\alpha = -\left[\frac{g}{L}\right]\theta \tag{10.12}$$

Now we recall the equations for the mass and spring. We had $F_x = -kx$ and $F_x = ma_x$. Combining these equations gives:

$$a_x = -\left[\frac{k}{m}\right]x \tag{10.13}$$

an equation which is similar in form to Eq. 10.12.

Now from Eq. 10.6 we found that ω^2 for the mass–spring system is k/m, namely the thing inside the square brackets in Eq. 10.13. It is sensible (and valid) to conclude that ω^2 for the simple pendulum is the thing inside the brackets in Eq. 10.12. So for the *pendulum* system we have:

$$\omega^2 = \frac{g}{L} \qquad \Longrightarrow \qquad \omega = \sqrt{\frac{g}{L}} \tag{10.14}$$

and then from $f = \omega/(2\pi)$ we have

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{10.15}$$

and using $T = 1/f$ we get the period of the pendulum,

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{10.16}$$

The result for f (or T) is surprising because of what is *not* in the formula. The mass of the pendulum bob is not there; as long as the mass is great enough so that air resistance is not significant the period is the same for *any* small bob attached to the end.

Secondly, the initial angle θ_0 does not appear. Does this mean that the period of a pendulum does not depend on how far back you pull it initially? Pretty much, yes. But we must recall that we are always working within the approximation that all the angles are "small"! If you pull the pendulum back by 60° there will be a significant (i.e. measurable) difference from the period you get by starting the pendulum at 5°.

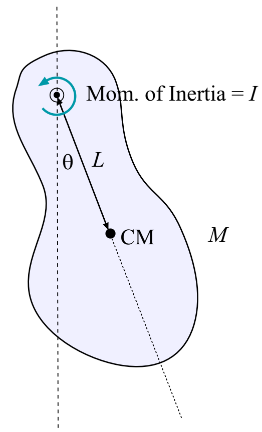

Figure 10.10: A "physical" pendulum. (What pendulum is *not* physical??) Object has moment of inertia I about the pivot; center of mass is a distance L from the pivot.

10.1.8 Physical Pendulum

For completeness, we give the results for a different kind of pendulum, but one which you may encounter in your problem sets or in the lab.

Suppose we suspend an object by a frictionless pivot so that its center of mass hangs below this pivot. Then we give the object a small displacement from the vertical position and let it oscillate back and forth.

Such an object is a kind of pendulum, but not a *simple* one because its mass is not concentrated at one point. People usually call it a **physical pendulum**, but I have yet to see a pendulum which is not "physical".

Hey, I just work here.

Anyway, when we write down the equations for the torque and angular acceleration of this object (similar to what we did above for the *simple* pendulum) we find that the frequency of the motion depends on the total mass M of the object, the moment of inertia of the object *about the pivot point* I and the distance L from the pivot to the center of mass of the object; see Fig. 10.10. Recall that to get the moment of inertia about points other than the center of mass the parallel axis theorem (Eq. 9.5) can be helpful.

The result for the frequency and period is

$$f = \frac{1}{2\pi}\sqrt{\frac{MgL}{I}} \qquad \text{and} \qquad T = 2\pi\sqrt{\frac{I}{MgL}} \tag{10.17}$$

10.2 Worked Examples

10.2.1 Harmonic Motion

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1. Atoms in a solid are not stationary, but vibrate about their equilibrium positions. Typically, the frequency of vibration is about $f = 2.0 \times 10^{12}\,\mathrm{Hz}$, and the amplitude is about $1.1 \times 10^{-11}\,\mathrm{m}$. For a typical atom, what is its (a) maximum speed and (b) maximum acceleration? [CJ7 10-15]

(a) The *angular* frequency of the atoms' oscillations is

$$\omega = 2\pi f = 2\pi(2.0 \times 10^{12}\,\mathrm{s}^{-1} = 1.26 \times 10^{13}\,\mathrm{s}^{-1}$$

so using Eq. 10.4 the maximum speed of atom is

$$v_{\mathrm{max}} = \omega A = (1.26 \times 10^{13}\,\mathrm{s}^{-1})(1.1 \times 10^{-11}\,\mathrm{m}) = 1.38 \times 10^{2}\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

(b) Using Eq. 10.5, the maximum acceleration of the atom is

$$a_{\mathrm{max}} = \omega^2 A = (1.26 \times 10^{13}\,\mathrm{s}^{-1})^2(1.1 \times 10^{-11}\,\mathrm{m}) = 1.66 \times 10^{15}\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

10.2.2 Mass–Spring System

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2. A $0.200\,\mathrm{kg}$ mass hanging from the end of a spring is set into motion. It is found that the mass bobs up and down through 10 cycles in 6.50 s. What is the force constant of the spring?

Find the period of the motion. Since 6.50 s was the time for *ten* cycles, we get :

$$T = \frac{(6.50\,\mathrm{s})}{(10.0\,\mathrm{cycle})} = 0.650\,\mathrm{s}$$

Then use Eq. 10.8 to solve for T:

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad \Longrightarrow \qquad T^2 = \frac{4\pi^2 m}{k} \qquad \Longrightarrow \qquad k = \frac{4\pi^2 m}{T^2}$$

Plug in stuff:

$$k = \frac{4\pi^2(0.200\,\mathrm{kg})}{(0.650\,\mathrm{s})^2} = 18.7\,\tfrac{\mathrm{kg}}{\mathrm{s}^2} = 18.7\,\tfrac{\mathrm{N}}{\mathrm{m}}$$

10.2.3 Simple Pendulum

---

3. What is the period of a simple pendulum which has a length of $3.00\,\mathrm{m}$?

Use Eq. 10.16 with $L = 3.00\,\mathrm{m}$. Get:

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{(3.00\,\mathrm{m})}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}}} = 3.48\,\mathrm{s}$$

The period of the pendulum is $3.48\,\mathrm{s}$.

---

4. What is the length of a simple pendulum which has a period of $1.00\,\mathrm{s}$?

Use Eq. 10.16; square both sides then solve for L. We get:

$$T^2 = 4\pi^2\frac{L}{g} \qquad \Longrightarrow \qquad L = \frac{T^2 g}{4\pi^2}$$

Plug in $T = 1.00\,\mathrm{s}$ and get:

$$L = \frac{(1.00\,\mathrm{s})^2(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{4\pi^2} = 0.248\,\mathrm{m}$$

A pendulum with a length of $0.248\,\mathrm{m} = 24.8\,\mathrm{cm}$ will have a period of $1.00\,\mathrm{s}$.

---

5. A simple pendulum of length $2.0\,\mathrm{m}$ is being swung on the surface of some strange planet. It makes 20 complete oscillations in $51.7\,\mathrm{s}$. What is the value of g on this planet?

To get the period of this pendulum, find the time per oscillation:

$$T = \frac{(51.7\,\mathrm{s})}{(20\,\mathrm{osc})} = 2.59\,\frac{\mathrm{s}}{\mathrm{osc}} = 2.59\,\mathrm{s}$$

Solve for g from Eq. 10.16:

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \Longrightarrow \qquad T^2 = 4\pi^2\frac{L}{g} \qquad \Longrightarrow \qquad g = \frac{4\pi^2 L}{T^2}$$

Plug in the numbers:

$$g = \frac{4\pi^2(2.0\,\mathrm{m})}{(2.59\,\mathrm{s})^2} = 11.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration of gravity on this planet is $11.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

Chapter 11

Waves I

11.1 The Important Stuff

11.1.1 Introduction

A **wave** (as we'll use the term in this chapter) is a disturbance in some continuous, deformable and otherwise uniform medium which travels over "long" distances while the little bits of the medium itself are moving over relatively small distances. Familiar examples of waves are the surface waves on water, the waves on a vibrating string and sound waves, which are travelling distortions in the density of air.

Light and electromagnetic waves in general have some properties in common with these types of waves, but electromagnetic waves are different in that they do not travel through any medium. Also, they arise from the electric and magnetic fields we'll study later on, and as such they are best left to that part of the course.

Wave phenomena occur around us all the time and if only for that reason it would be important to study them. But it turns out that when we look at the behavior of matter on the smallest scales, the basic "particles" in nature *don't* behave like particles following Newton's laws as presented in this course; rather, they are waves.

In the end, it's all waves.

Waves are characterized by the direction in which the little bits of the medium move in relation to the direction of motion of the wave itself.

For a wave on a taut string the elements of the string move *perpendicularly* to the direction of propagation of the wave, as shown in Fig. — (a). Such a wave is called a **transverse** wave. These waves are the easiest to visualize.

For others, the motion of the medium is *along* the direction of propagation of the wave. An example of this (one that can be *seen*, at least) is that of a Slinky which has been tapped along its length. The disturbance which one sees travelling down the Slinky is a region where

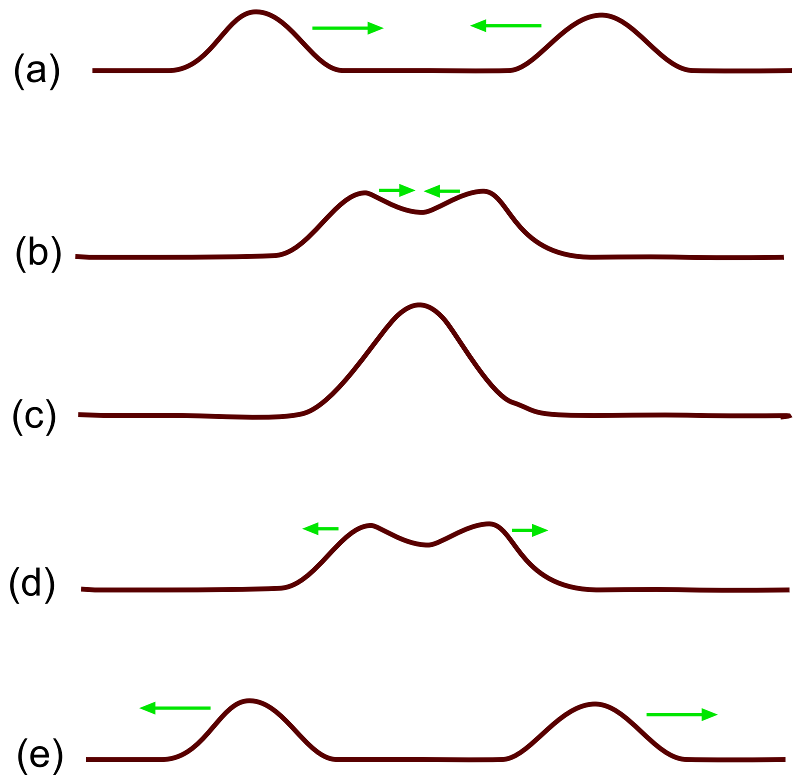

Figure 11.1: Two "positive" pulses are put on a string; they travel towards one another, add *constructively* and then continue on as if nothing had happened!

the coils are slightly compressed. This type of wave is called a **longitudinal** wave.

The waves (and the media through which they travel) which we'll study will have the property that the disturbance will propagate unchanged.

11.1.2 Principle of Superposition

Another property of the waves (and media) which we'll study is that individual waves "add together". What we mean by this is that if we have two sources of waves the wave that is present when both sources operate at once is the *sum* of the waves that are present when the sources are operating individually. We do mean the literal *sum* here; we add the displacements of the medium at each point of the medium.

The idea of adding waves together is called the *principle of superposition* and it is obeyed by all the media and waves that we will consider.

An example is given in Fig. 11.1, something which you and a friend could demonstrate with a rope, but you'd have to observe it very quickly! Here two pulses are put onto a rope

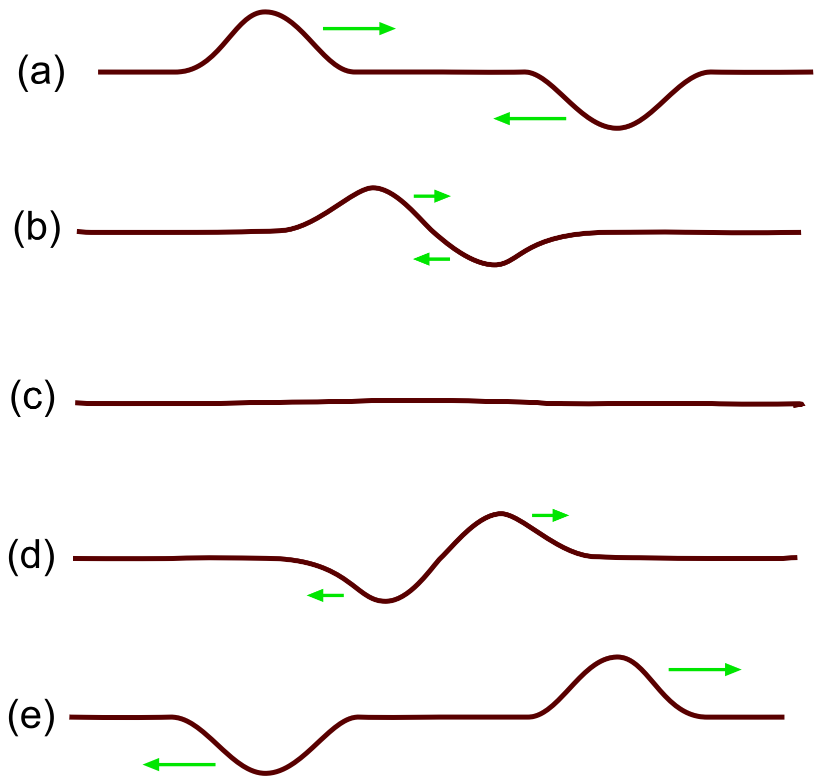

Figure 11.2: “Positive” and “negative” pulses of similar shape are created on a string and travel toward each other. When they overlap the pulses cancel such that the rope is nearly flat at some point. Then the pulses emerge and keep on travelling.

by some sources, like the hands of the people holding the rope. The pulses travel toward one another meet and continue onward after they meet. The displacements of the rope making the pulses are in the same direction for the two pulses. When the pulses meet at the center their displacements add together to give a larger pulse. Somehow the rope “knows” that it contains two individual pulses so that the pulses again separate and keep moving in their original directions!

I kid you not, this really happens.

When waves combine so that the resulting wave is “larger” than the individual waves, we say that **constructive interference** is taking place.

Even stranger is when the two people holding the rope create travelling pulses with opposite displacements, i.e. for one pulse the string disturbance goes “up” while for the other it goes “down”, as shown in Fig. 11.2. When the pulses begin to overlap the sum of the waves has a smaller size than the individual wave and we say that **destructive interference** is taking place. When the waves precisely overlap the string is very nearly flat! (We assume the two pulses had similar shapes.) Nevertheless, an instant later the two

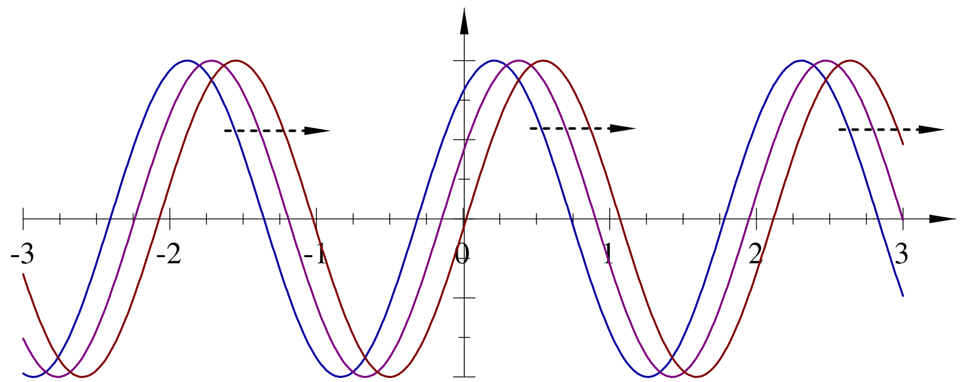

Figure 11.3: A harmonic wave is a sinusoidal pattern that travels to the left or right.

pulses emerge and continue on their separate ways.

How can this be? If the string is momentarily flat, haven't the two waves been "killed"? Then why do the pulses reappear? What you don't see in Fig. 11.2(c) is the instantaneous *velocity* of the parts of the string and these are *not* zero. The "memory" of the shapes is contained in that movement and when we wait a little longer we get the two pulses back again.

11.1.3 Harmonic Waves

Things could get mighty complicated if we were to consider waves of any shape whatsoever. It turns out that if we focus on a particular kind of wave with a repeating pattern of displacement we will know about all we need to know about waves. Some of the reasons are "beyond the scope of the course" as they say, but it's also true that many wave sources will produce a repeating pattern and then our results will be useful for describing the real world.

The kind of wave we want to study is shown in Fig. 11.3. It is sinusoidal in shape and is essentially infinite in length. And since it is a wave, it is *moving* in some direction, either to the left or to the right with some speed v.

Can you visualize that? I hope so. . . if only I could put an *animated* picture of the wave onto this page! Maybe in the future the pages of books will be able to show *moving* pictures. For now we're stuck with paper and still pictures. So you'll have to imagine that the pattern in Fig. 11.3. is moving to the right.

Such a wave is called a **harmonic wave**.

Again, using your imagination think of the moving sinusoidal pattern as a wave on a string and consider the motion of *any single point on the string*. A single point just moves up and down, much like a harmonic oscillator in the previous chapter; see Fig. 11.4. In fact for the harmonic wave, the motion of a single point is *exactly* that of a harmonic oscillator;

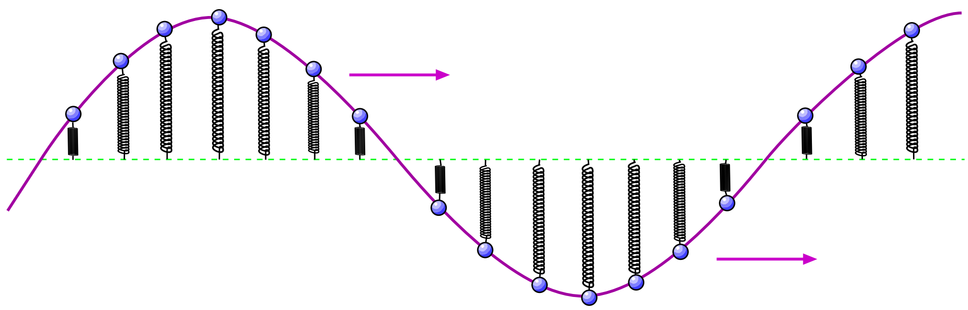

Figure 11.4: The individual points on the harmonic wave act like harmonic oscillators, moving up and down.

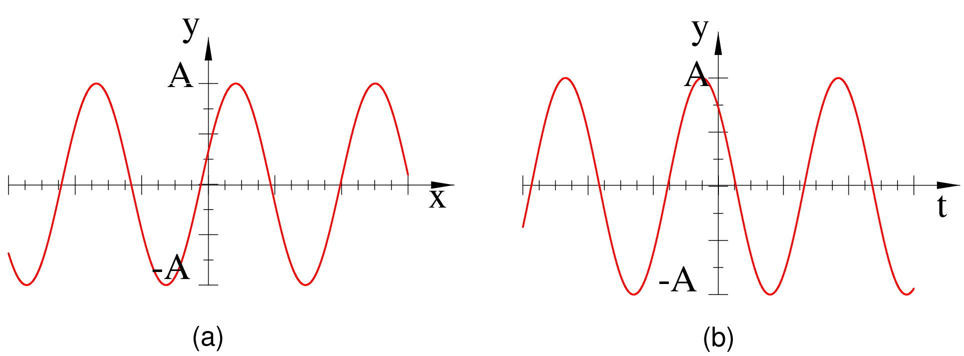

Figure 11.5: (a) At a fixed time t the displacement y has a sinusoidal dependence on the coordinate x. (b) At a fixed *location* x the displacement has a sinusoidal dependence on the time t.

it is sinusoidal in its *time* dependence also.

The harmonic wave is sinusoidal in a double sense: Freeze the time and the wave is sinusoidal in x. Fix the location x and the wave is sinusoidal in the time t. This is shown in Fig. 11.5.

Damn, I wish I could put animated pictures in a book.

Two important numbers which characterize a harmonic wave are the wavelength and the frequency. In Fig. 11.6 we show a picture of a harmonic wave "frozen" at some particular time t. The length of string from any place on the wave to where it starts to repeat is the **wavelength** of the wave. It is usually given the symbol λ and is measured in meters (since it is a length!).

Next we think of the motion of an individual bit of the string (like the individual elements shown in Fig. 11.4). A plot of its displacement versus time might look like Fig. 11.7. The string element oscillates up and down and therefore its motion has a *frequency*, just like the

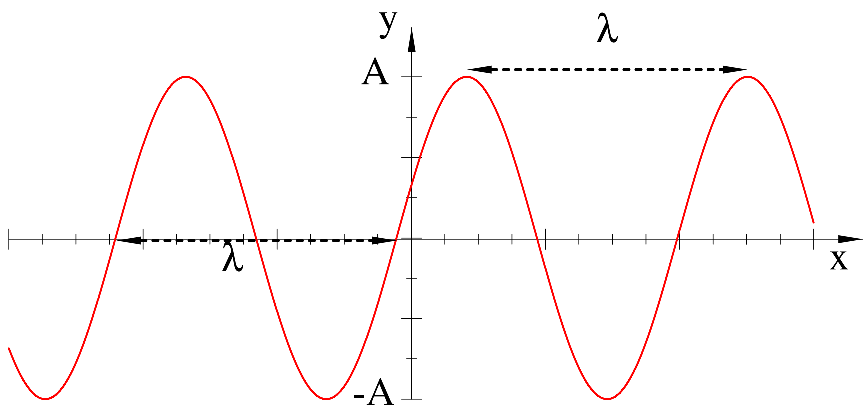

Figure 11.6: When we "freeze" the time t, the wavelength λ is the length along the string for one full cycle of the displacement.

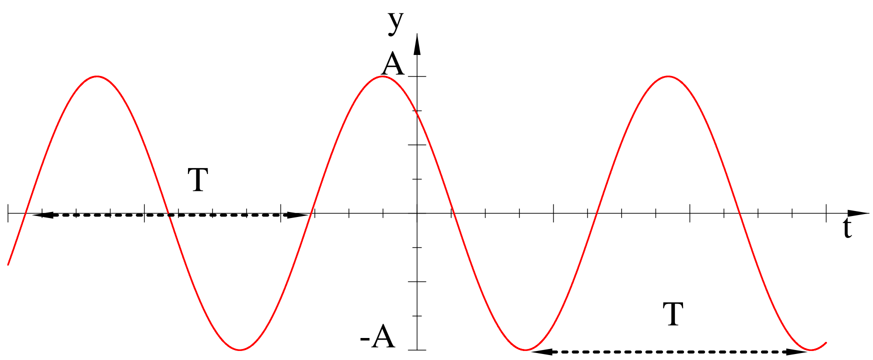

Figure 11.7: When we choose a bit of the string at a particular x, the period T is the time needed for one full cycle of the displacement. The frequency of the wave is given by $f = 1/T$.

oscillators in the last chapter. This is what we mean by the frequency of the *wave*. The frequency (f) is measured in Hz=$\frac{\text{cycle}}{\text{s}}$.

And now we want to think about the space and time dependence of the wave simultaneously. If you can really visualize the wave motion (and it is a bit tricky) you can see that in the time T (the time in which one bit of the string moves through a full cycle) the wave *moves forward by one wavelength*, i.e. in a time T the waves moves forward a distance λ. That means that the speed of the wave (v) is given by:

$$v = \frac{\lambda}{T}$$

But since $T = 1/f$ this gives us

$$v = \frac{\lambda}{(1/f)} = \lambda f \ ,$$

that is,

$$\lambda f = v \tag{11.1}$$

11.1.4 Waves on a String

The speed of waves on a string depends on the tension (called F here) and the "bulkiness" of the string material, which is given by the mass per unit length of the string. For a string of length L and mass m under tension F it is given by:

$$v = \sqrt{\frac{F}{(m/L)}} = \sqrt{\frac{F}{\mu}} \tag{11.2}$$

where $\mu = m/L$.

11.1.5 Sound Waves

A sound wave is a longitudinal wave in the *density* of some medium; most often we consider the atmosphere as the medium but it could be a liquid or solid. Fluctuations in the density of the medium occur because of the tiny motions of the elements of the medium (the atoms and molecules) *along* the direction of motion of the wave.

Because of the collective motions of the molecules there are regions where the density is slightly greater or less than the "normal" density of the medium and it is these regions of greater or lesser density which from the disturbance which travels, i.e. the sound wave. An exaggerated picture is shown in Fig. 11.8. Again, the problem with such still pictures is that they don't show the *motion* of the wave or how the high/low density regions are formed by the motions of the little bits of the medium.

I wish I knew how to put animated pictures in these notes! *Then* it would look good.

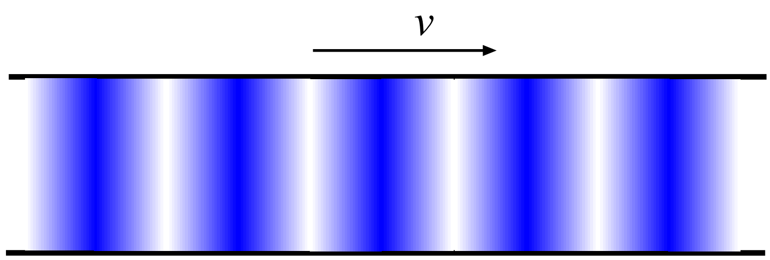

Figure 11.8: Sound wave travels down air air-filled pipe with speed v.

Damn.

Anyway, the speed of sound is strongly dependent on the type of medium in which the wave travels. Even for sound waves in air at normal pressure the speed depends significantly on the temperature of the air: In air at at 1 atm of pressure and at 20°C the speed is 343 $\frac{\mathrm{m}}{\mathrm{s}}$ but at a temperature of 0°C it is 331 $\frac{\mathrm{m}}{\mathrm{s}}$.

There is actually a formula for the speed of sound waves in a gas. It involves some quantities not covered yet in these notes but it is useful to write it down here. Suppose the *absolute* (Kelvin) temperature of the gas is T and the mass of one of its molecules is m. There is a number important is the study of the thermal properties matter called the *Boltzmann constant*, k which is given by:

$$k = 1.38 \times 10^{-23}\ \frac{\mathrm{J}}{\mathrm{K}}$$

Finally there is a unitless number symbolized by γ which is characteristic of the type of molecules in the gas and some other factors; for monatomic gases under "normal" (room temperature) conditions it is $\frac{5}{3}$; for diatomic gasses (like those in the atmosphere, N_2 and O_2) it is $\frac{7}{5}$. Anyway, with all of this, the speed of sound is given by

$$v = \sqrt{\frac{\gamma k T}{m}} \tag{11.3}$$

11.1.6 Sound Intensity

The two important aspects of a sound wave are its frequency (if it is a harmonic wave) and its amplitude (i.e. its loudness). Since in a sound wave the particles of the medium oscillate back and forth (on average) along the direction of propagation we could talk about a **displacement amplitude** for the sound wave. Likewise, since the wave is formed of pressure variations along the direction of propagation, one could also speak of a **pressure amplitude** for the wave. These two type of amplitudes are related, but we won't discuss them further; refer to more complete physics texts.

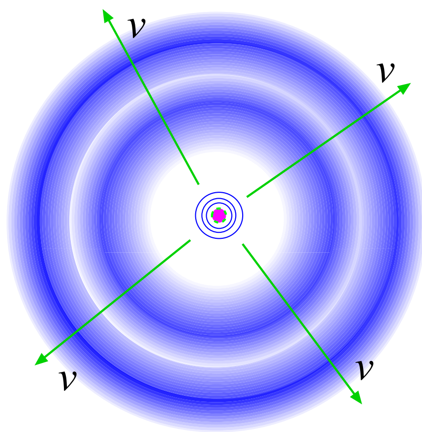

Figure 11.9: Spherical wave.

It is more useful to describe the loudness of a sound wave in terms of the rate at which it transports *energy*. At a large distance from a source of sound, the wave maxima are close to a set of equally spaced planes moving forward at speed v. If we consider a planar surface of area A through the wave passes then in a time t the wave will transfer an amount of energy E; E is clearly proportional to both A and t, so dividing them out gives a measure of the strength of the sound wave, its **intensity**:

$$I = \frac{E}{At} \tag{11.4}$$

The intensity I has units of $\frac{\mathrm{J}}{\mathrm{m^2 \cdot s}}$, or $\frac{\mathrm{W}}{\mathrm{m^2}}$.

The human ear can detect sounds with a large range of intensities, from $10^{-12}\,\frac{\mathrm{W}}{\mathrm{m^2}}$ (the threshold of hearing) to $10\,\frac{\mathrm{W}}{\mathrm{m^2}}$, for which the sound wave is intense enough to be harmful. It is useful to have a measure of intensity which gives the order of magnitude of the intensity, and such a measure is given by the **intensity level**, defined by:

$$\beta = (10\,\mathrm{dB}) \log_{10}\left(\frac{I}{I_0}\right) \tag{11.5}$$

where $I_0 = 10^{-12}\,\frac{\mathrm{W}}{\mathrm{m^2}}$. The intensity level β has units of decibels, or dB.

Sometimes, a useful approximation in dealing with sound waves is to imagine that the wave was created by a source that makes a wave go out radially, that is, equally is all directions. This is the approximation of an **isotropic source**, and the wave from such a source is illustrated in Fig. 11.9.

Such a source will put a certain amount of power P into the generation of the wave and this energy is transmitted outward in all directions. If we consider a spherical surface of radius r centered on the source then the rate at which energy crosses this surface is also P. The energy flux at this distance can then be found by taking the total power crossing the surface (P) and dividing by the total area of that surface ($4\pi r^2$). Then the intensity of sound waves at distance from an isotropic source of power P is

$$I = \frac{P}{4\pi r^2} \tag{11.6}$$

11.1.7 The Doppler Effect

An interesting effect occurs when a source of sound (isotropic, for simplicity) is *in motion* through the medium through which the sound travels (usually the air, which we take to be "stationary"). In Fig. — we show a source in motion toward an observer. The source has speed v_s and here the observer is at rest with respect to the air.

We see that since the waves were emitted *from different points* (due to the motion of the source) the wave maxima (indicated with the circles) are bunched up in front of the source, that is, more bunched up than if the source were standing still. Likewise, they are more separated behind the source.

In this case, the observer will receive waves which are travelling at the usual speed but which effectively have a shorter wavelength. Then the frequency of the wave must be *larger* than if the source were standing still. One can show that the observer would hear a frequency f_o given by

$$f_o = f_s \left(\frac{1}{1 - \frac{v_s}{v}} \right)$$

where v is the speed of sound (in air), f_s is the frequency of the source and v_s is the speed of the source toward the observer.

Conversely, if the observer had been behind the source, s/he would hear a frequency *lower* than if the source were standing still and it would be given by

$$f_o = f_s \left(\frac{1}{1 + \frac{v_s}{v}} \right)$$

The two formulae can be combined as:

$$f_o = f_s \left(\frac{1}{1 \mp \frac{v_s}{v}} \right) \tag{11.7}$$

where we take the bottom sign for motion of source *toward* the observer

There will also be an change in the frequency received from a source is the source is stationary but the *observer* is in motion. Fig. — shows an observer running toward a source of sound. In this case, the spacing of the wave maxima has its normal value, but the observer encounters the wave maxima at a greater rate because of his motion. In effect, the *speed* of the waves is greater and so again the observer hears a larger frequency.

If the observer had been running away from the source he would have heard a lower frequency.

The result for the two cases can be combined into one formula as:

$$f_o = f_s \left(1 \pm \frac{v_o}{v}\right) \tag{11.8}$$

where as before we take the bottom sign for motion of the observer toward the source.

If *both* the source and observer are in motion then we must have to use the most general formula, which is basically a combination of Eqs. 11.7 and 11.8:

$$f_o = f_s \left(\frac{1 \pm \frac{v_o}{v}}{1 \mp \frac{v_s}{v}}\right) \tag{11.9}$$

where the choice of sign is the top one for "toward" and the bottom one for "away".

11.2 Worked Examples

11.2.1 Harmonic Waves

1. A harmonic wave on a string has a speed of $220\,\frac{\text{m}}{\text{s}}$ and a wavelength of $0.720\,\text{m}$. What is the frequency of the wave?

Use Eq. 11.1 and solve for f:

$$\lambda f = v \qquad \Longrightarrow \qquad f = \frac{v}{\lambda}$$

Substitute for v and λ and get:

$$f = \frac{(220\,\frac{\text{m}}{\text{s}})}{(0.720\,\text{m})} = 306\,\text{Hz}$$

The frequency of the wave is $306\,\text{Hz}$.

Note that even though the units come out as "$\frac{1}{\text{s}}$" here, a *frequency* should be expressed in units of $\frac{\text{cycle}}{\text{s}} = \text{Hz}$.

2. What is the wavelength of a harmonic wave on a string if the speed of the waves is $160\,\frac{\text{m}}{\text{s}}$ and the frequency of the wave is $220\,\text{Hz}$?

Use Eq. 11.1 and solve for λ; plug in the numbers and get

$$\lambda = \frac{v}{f} = \frac{(160\,\frac{\text{m}}{\text{s}})}{(220\frac{\text{cycle}}{\text{s}}} = 0.727\,\text{m}$$

11.2.2 Waves on a String

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3. The linear density of the A string on a violin is $7.4 \times 10^{-4}\,\text{kg/m}$. A wave on the string has a frequency of $440\,\text{Hz}$ and a wavelength of $65\,\text{cm}$. What is the tension in the string? [CJ6 16-13]

We are given the values of λ and f for this string wave so from Eq. 11.1 we ca get the speed of the wave:

$$v = \lambda f = (0.650\,\text{cm})(440\,\text{s}^{-1}) = 286\,\frac{\text{m}}{\text{s}}$$

Eq. 11.2 relates the wave speed v to the tension F and mass density μ; solve for F:

$$v = \sqrt{\frac{F}{\mu}} \qquad \Longrightarrow \qquad v^2 = \frac{F}{\mu} \qquad \Longrightarrow \qquad F = \mu v^2$$

Plug in:

$$F = (7.4 \times 10^{-4}\,\frac{\text{kg}}{\text{m}})(286\,\frac{\text{m}}{\text{s}})^2 = 60.5\,\text{N}$$

11.2.3 Sound Waves